136

THE QUARTERLY JOURNAL OF

MECHANICS AND APPLIED MATHEMATICS

VOLUME XI PART 4

NOVEMBER 1958

OXFORD
AT THE CLARENDON PRESS
1958

Price 18*s. net*

PRINTED IN GREAT BRITAIN BY VIVIAN RIDLER AT THE UNIVERSITY PRESS, OXFORD

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DETERMINATION OF STABILITY BY LINEAR APPROXIMATION OF A PERIODIC SOLUTION OF A SYSTEM OF DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS RIGHT-HAND SIDES†

By M. A. AIZERMAN *and* F. R. GANTMACHER
(*Moscow*)

[Received 28 November 1957]

SUMMARY

In the works of Liapounoff (**1**) the problem of stability of a periodic solution $z_i = \tilde{z}_i(t)$ (with period τ) of a system of differential equations

$$\frac{dz_i}{dt} = f_i(z_1,\ldots,z_n,t) \quad [f_i(z_1,\ldots,z_n t+\tau) \equiv f_i(z_1,\ldots,z_n,t)] \tag{0.1}$$

was reduced in the principal (non-critical) cases to the problem of stability of the zero-solution $x_i = 0$ of the system of linear approximation‡

$$\frac{dx_i}{dt} = \sum_k \left(\frac{\partial f_i}{\partial z_k}\right)_{z=\tilde{z}(t)} x_k. \tag{0.2}$$

This reduction is applicable when the right-hand sides of equations (0.1) are analytic functions, or functions continuous in the neighbourhood of the integral curve and differentiable with respect to $z_1,\ldots, z_n$, uniformly in regard to t at the points of this curve.

It is the purpose of this paper to analyse the stability of a periodic solution $z_i = \tilde{z}_i(t)$ $[\tilde{z}_i(t+\tau) \equiv \tilde{z}_i(t)]$ of system (0.1) when the right-hand sides f_i are discontinuous at some points of the integral curve $z_i = \tilde{z}_i(t)$. An explanation of the meaning of the linear approximation for this 'discontinuous' case is offered, and theorems, analogous to those of Liapounoff on stability by linear approximation in the continuous case, are presented.

1. Formulation of the problem and principal theorems

In order to obtain conditions to which the right-hand sides of the system of equations (0.1) should conform, consider in an $(n+1)$-dimensional space $z_1,\ldots, z_n$, t a curvilinear cylinder C whose axis is the integral curve of the undisturbed motion $z_i = \tilde{z}_i(t)$. Let an infinite sequence of hypersurfaces (discontinuity surfaces)§

$$F_\alpha(z_1,\ldots,z_n,t) = 0 \tag{1.1}$$

† This article has been published in Russian in *Prikl. Mat. Mek.* **21** (1957).

‡ Here and elsewhere latin indices i, j, k have values 1, 2,..., n, and index α values 1, 2,..., ∞.

§ Henceforth, instead of 'hypersurface' and 'hyperplane' we will write 'surface' and 'plane'.

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divide the cylinder C into domains H_α, and let the integral curve $z_i = \tilde{z}_i(t)$ intersect the discontinuity surface (1.1) at points M_α when $t = t_\alpha$, crossing over from the 'negative' to the 'positive' face of this surface (Fig. 1).

It is supposed that the right-hand sides of equations (0.1) conform to the following conditions:

(1) The functions f_i are periodic in respect to t, with a period τ.

(2) The functions f_i are continuous in each domain H_α, including the

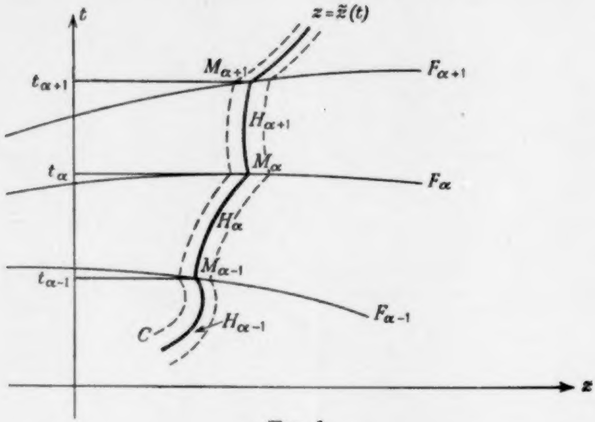

FIG. 1

boundaries $F_{\alpha-1} = 0$ and $F_\alpha = 0$, and on passing through the surface of $F_\alpha = 0$, they have only discontinuities of the first type.

(3) The functions f_i in the domain H_α are differentiable with respect to $z_1,\ldots, z_n$ at the points of the integral curve $z_i = \tilde{z}_i(t)$, and uniformly differentiable with respect to t, i.e.

$$f_i(z_1,\ldots,z_n,t) = f_i(\tilde{z}_1,\ldots,\tilde{z}_n,t)+\sum_j \left(\frac{\partial f_i}{\partial z_j}\right)_{z=\tilde{z}(t)}(z_j-\tilde{z}_j)+o(\rho), \tag{1.2}$$

where $\rho = \left[\sum_j (z_j-\tilde{z}_j)^2\right]^{\frac{1}{2}}$ and $\frac{o(\rho)}{\rho} \to 0$, as $\rho \to 0$,

uniformly in regard to t in each interval $t_{\alpha-1} \leqslant t \leqslant t_\alpha$.

(4) The conditions of existence and uniqueness of the solution of system (0.1) for given initial conditions are fulfilled in every domain H_α, and the integral curves depend continuously on the initial conditions. As usual, it is supposed that the integral curves of the system (0.1) are continuous all over C, including the points of intersection with the discontinuity surfaces of the right-hand sides f_i.

As regards the discontinuity surfaces, we make the following assumptions:

(5) The surfaces $F_\alpha = 0$ are continuous, and at points M_α are smooth.

(6) The derivatives $(dF_\alpha/dt)^-$ and $(dF_\alpha/dt)^+$ along the integral curve $z_i = \tilde{z}_i(t)$ are non-zero and are of the same sign.†

(7) Translation of the family of surfaces (1.1) along the axis over a distance equal to period τ transfers it into itself.

Along with system (0.1) consider the system of linear differential equations

$$\frac{dx_i}{dt} = \sum_k \left(\frac{\partial f_i}{\partial z_k}\right)_{z=\tilde{z}(t)} x_k, \tag{1.3}$$

and assume that functions x_i in each interval $t_{\alpha-1}+0 \leqslant t \leqslant t_\alpha - 0$ are continuous and satisfy equations (1.3) and at $t = t_\alpha$ have discontinuities determined by the formulae

$$x_i^+ - x_i^- = \xi_i \sum_k h_k^- x_k^- \tag{1.4.1}$$

or

$$x_i^+ - x_i^- = \xi_i \sum_k h_k^+ x_k^+, \tag{1.4.2}$$

where ξ_i is the value of the jump of function f_i at point M_α when crossing from the negative to the positive side of surface $F_\alpha = 0$ and‡

$$h_k^\pm = \left[\frac{\partial F_\alpha}{\partial z_k} \Big/ \left(\frac{dF_\alpha}{dt}\right)^\pm\right]_{M_\alpha}.$$

We will prove in the next paragraph the equivalence of conditions (1.4.1) and (1.4.2). These conditions will be called the *discontinuity conditions*.

The linear differential system determined by the linear differential equations (1.3) and by the linear discontinuity conditions (1.4) will be called the *linear approximation* of the initial non-linear system (0.1).

The significance of the linear approximation thus introduced will be seen from the following theorem.

THEOREM 1. *If the zero-solution of a system of linear approximation is asymptotically stable, then a periodic solution $z_i = \tilde{z}_i(t)$ of the initial non-linear system* (0.1) *is also asymptotically stable.*

The characteristic equation of linear approximation is defined in the same way as in the continuous case. Denote the fundamental matrix of

† Here
$$\frac{dF_\alpha}{dt} = \left[\sum_k \frac{\partial F_\alpha}{\partial z_k} f_k + \frac{\partial F_\alpha}{\partial t}\right]_{z=\tilde{z}(t)},$$
and indices $-$ and $+$ define values taken at $t = t_\alpha - 0$ and $t = t_\alpha + 0$, i.e. before and after the point of discontinuity.

‡ Quantities ξ_i and $h_k^\pm$ depend also on index α, but for the sake of clearness it is omitted.

the linear approximation by $X(t)$. Its columns are linearly-independent solutions of the systems (1.3)+(1.4). Then the following identity holds:

$$X(t+\tau) = X(t)U, \tag{1.6}$$

where U is a certain non-singular constant matrix. The characteristic equation $\det(U-\rho E) = 0$ of the matrix U (here E is a unit matrix) is called the *characteristic equation of linear approximation.*

THEOREM 2. *If the absolute value of at least one of the roots of the characteristic equation of linear approximation is larger than unity, then the periodic solution* $z_i = \tilde{z}_i(t)$ *of system* (0.1) *is unstable.*

2. Preliminary remarks

1°. The peculiarity of Theorems 1 and 2 lies in the fact that the analysis of the stability of continuous trajectories of system (0.1) is replaced by the analysis of the stability of discontinuous trajectories of the linear approximation (1.3)+(1.4). It is assumed that for these discontinuous trajectories the definitions of stability and asymptotic stability are the same as those introduced by Liapounoff for continuous trajectories.

2°. All the equations cited above were written in scalar form in order to be able to formulate the generalization of Liapounoff's theorem, in the same terms and notation as in Liapounoff (**1**). In order to prove these theorems it is more convenient to state the basic relations in matrix form. For this purpose consider columns z, f, x, ξ, h^-, and h^+, composed of elements $z_i, f_i, x_i, \xi_i, h_i^-$, and h_i^+ respectively, and also square matrices†

$$\frac{\partial f}{\partial z} = \left\|\frac{\partial f_i}{\partial z_k}\right\|, \qquad B^- = \|\xi_i h_k^-\| = \xi h^{-\prime}, \qquad B^+ = \|\xi_i h_k^+\| = \xi h^{+\prime}.$$

Now we can write the system of initial equations (0.1) and the linear approximation (1.3)+(1.4) in matrix form:

$$\frac{dz}{dt} = f(z,t), \tag{2.1}$$

$$\frac{dx}{dt} = \left(\frac{\partial f}{\partial z}\right)_{z=\tilde{z}(t)} x, \tag{2.2}$$

$$x^+ - x^- = B^- x^- \tag{2.3.1}$$

or

$$x^+ - x^- = B^+ x^+. \tag{2.3.2}$$

3°. Let us use the matrix form of notation in order to prove the equivalence of the discontinuity conditions (2.3.1) and (2.3.2), or (1.4.1)

† Here and in what follows the accent ′ denotes the transposed matrix.

and (1.4.2), which are the same. Equalities (2.3.1) and (2.3.2) may be written in the form:

$$x^+ = (E+B^-)x^-, \qquad x^+ = (E-B^+)^{-1}x^-. \tag{2.3.3}$$

We will show that $E+B^- = (E-B^+)^{-1}$. Indeed, this equality can be reduced to equality $B^- - B^+ - B\ B^+ = 0$, the correctness of which follows from

$$B^-B^+ = \xi h^{-\prime}\xi h^{+\prime} = (h^{-\prime}\xi)\xi h^{+\prime} = cB^+ \quad (c = h^{-\prime}\xi)$$

and

$$B^- - B^+ = \xi\left[\frac{\partial F_\alpha/\partial z}{(dF_\alpha/dt)^-} - \frac{\partial F_\alpha/\partial z}{(dF_\alpha/dt)^+}\right]' = cB^+.$$

Here $\partial F_\alpha/\partial z$, i.e. the column of elements $\partial F_\alpha/\partial z_i$, is the gradient of function F_α. From the equivalence of conditions (1.4.1) and (1.4.2) it follows that whenever $t = t_\alpha$, we have

$$\sum_k h_k^- x_k^- = \sum_k h_k^+ x_k^+. \tag{2.4}$$

Now consider the matrix

$$S = E+B^- = (E-B^+)^{-1}.$$

We will write the conditions of discontinuity in the form

$$x^+ = Sx^-. \tag{2.5}$$

Let us now examine the structure of matrix S and the transformation (2.5) that it generates. Let K denote the intersection of the plane tangential to the surface of discontinuity $F_\alpha = 0$ (at the point M_α) with the plane $t = t_\alpha$. The vectors in the plane $t = t_\alpha$, parallel to K satisfy the equation $h^{-\prime}x = 0$.

Let
$$\gamma = \frac{(dF_\alpha/dt)^+_{M_\alpha}}{(dF_\alpha/dt)^-_{M_\alpha}} = 1+h^{-\prime}\xi = 1+c.$$

In accordance with condition (6) of section 1, the quantity $\gamma > 0$.

Let us examine the three cases separately.

Case 1. $\gamma \neq 1$. In this case $c = h^{-\prime}\xi \neq 0$ and ξ does not belong to K. For any vector x we have a unique expansion $x = x_\xi + x_k$ ($x_\xi \| \xi$, $x_k \in K$).

Since $S\xi = \xi + \xi(h^{-\prime}\xi) = \gamma\xi$ and $x_\xi \| \xi$ we have

$$Sx_\xi = \gamma x_\xi.$$

On the other hand, $\quad Sx_k = x_k + \xi(h^{-\prime}x_k) = x_k.$

Hence, the transformation (2.5) does not change vectors parallel to K, but vectors parallel to ξ are multiplied by γ. Between the components of vectors x^- and x^+ there exists a relation

$$x_\xi^+ = \gamma x_\xi^-, \qquad x_k^+ = x_k^-.$$

It follows that matrix S has simple elementary divisors and the characteristic root γ is simple and 1 is of $(n-1)$-multiplicity.

Case 2. $\gamma = 1$, $\xi \neq 0$, $h \neq 0$. Here $\xi \in K$. All characteristic roots of the matrix S are equal to 1, but $S \neq E$. Matrix S has one elementary divisor of the second degree, all the others being of the first.

Case 3. $\gamma = 1$ and $\xi = 0$ or $h = 0$. In this case $S = E$ and $x^+ = x^-$. Therefore all integral curves of the linear approximation are continuous at $t = t_\alpha$ only when

the right-hand sides of the differential equations (0.1) are continuous at point M_α of the integral curve $z_i = \tilde{z}_i(t)$, or if at this point the discontinuity surface is tangential to plane $t = t_\alpha$.

4°. Let us introduce the discontinuous functions

$$h_k(t) = \left[\frac{\partial F_\alpha/\partial z_k}{dF_\alpha/dt}\right]_{z=\tilde{z}(t)}.$$

It follows from (2.4) that the function $\sum\limits_k h_k(t)x_k(t)$ is continuous. Therefore the system of linear approximation may be written

$$Dx_i = \sum_k \left[\left(\frac{\partial f_i}{\partial z_k}\right)_{z=\tilde{z}(t)} + \sum_\alpha \xi_i h_k(t)\,\delta(t-t_\alpha)\right]x_k \tag{2.6}$$

where $\delta(t)$ is the Dirac-function and Dx_i the derivative of function x_i with regard to δ-terms.†

On the other hand, it can be shown by simple calculations, that the expression in square brackets is a generalized derivative with respect to z_k of the right-hand side f_i (i.e. a derivative with regard to δ-terms)

$$(D_k f_i)_{z=\tilde{z}(t)} = \left(\frac{\partial f_i}{\partial z_k}\right)_{z=\tilde{z}(t)} + \sum_\alpha \xi_i h_k(t)\delta(t-t_\alpha). \tag{2.7}$$

Thus, the system of linear approximation (1.3)+(1.4) can be written

$$Dx_i = \sum_k (D_k f_i)_{z=\tilde{z}(t)} x_k \tag{2.8}$$

which differs from the linear approximation (0.2) with continuous right-hand sides only‡ in having the ordinary derivatives $\partial f_i/\partial z_k$ *replaced by generalized* derivatives $D_k f_i$, and the product of a continuous function $u(t)$ by $\delta(t-t_\alpha)$ is equal to $u(t_\alpha)\delta(t-t_\alpha)$. It should be remembered that ξ_i and $h_k(t)$ depend on index α, which, as has already been pointed out, is omitted.

5°. We will now consider the deviation

$$x = z-\tilde{z}(t) \tag{2.9}$$

and rewrite equation (2.1) in terms of deviations

$$\frac{dx}{dt} = p(x,t) \qquad [p(x,t) = f(\tilde{z}(t)+x,t)-f(\tilde{z}(t),t)]. \tag{2.10}$$

† We proceed here from the fact that the derivative of any function $g(t)$ that is discontinuous at $t = t_\alpha$ with a jump η_α may be expressed with regard to δ-terms by a sum

$$\frac{dg}{dt} + \sum_\alpha \eta_\alpha \delta(t-t_\alpha).$$

‡ As applied to a particular case of a system, differing from a linear system with constant coefficients by having on the right-hand side one non-linear function of one argument of the type sign z, in (2), the linear approximation was composed of δ-functions in a form which is a particular case of equation (2.6). The author based his work, in essence, on the theorem of Liapounoff, although the application of it in the case of discontinuous systems was not proved.

The equations of the discontinuity surfaces will now be

$$P_\alpha(x,t) = 0 \qquad [P_\alpha(x,t) \equiv F_\alpha(\tilde{z}(t)+x,t)]. \tag{2.11}$$

In $(n+1)$-dimensional space (x,t) the discontinuity surface (2.11) is continuous, and so is the surface $F_\alpha = 0$ in the space (z,t), but in contrast with the surface $F_\alpha = 0$ in the space (z,t), the surface (2.11) in the space (x,t) is not smooth and has an edge at the line of intersection with plane $t = t_\alpha$.

Because of the second term in formula (2.10) the function $p(x,t)$ has

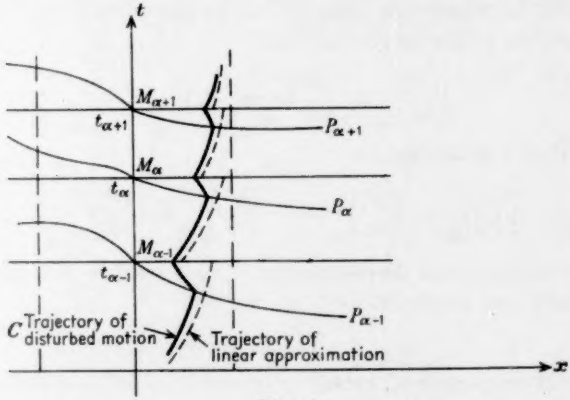

FIG. 2

a discontinuity not only on surfaces $P_\alpha = 0$, but also on planes $t = t_\alpha$. The solutions of systems (2.10) are also continuous, as are the solutions of the initial system (0.1). The space (z,t) is transformed into the space (x,t) (Fig. 2) and each plane $t =$ const. is translated upon itself. The point of intersection of this plane with the curve $z_i = \tilde{z}_i(t)$ transforms into a point of the t-axis. Now, the t-axis becomes the axis of cylinder C in space (x,t). The surfaces $P_\alpha = 0$ and planes $t = t_\alpha$ divide the cylinder C into *central* and *angular* domains. The latter are located between the surfaces $P_\alpha = 0$ and the corresponding planes $t = t_\alpha$. We will name the angular domains *upper* or *lower*, depending upon whether the boundary surface $P_\alpha = 0$ of the domains lies over or under the corresponding plane $t = t_\alpha$, respectively. At each point of the t-axis, differing from points M_α the function $p(x,t)$ is zero. At the point M_α of the t-axis $(t = t_\alpha)$ the function $p(x,t)$ may have three values: zero when approaching from the central region, ξ and $-\xi$ when approaching, respectively, from the lower or upper angular domains.

6^0. Now we can give a simple geometrical interpretation of the dis-

continuity conditions. Let us suppose, for instance, that an integral curve of the non-linear system (2.10) intersects the lower angular domain, adjoining the plane $t = t_\alpha$. We will choose the trajectory close to the t-axis. Thus, we may consider that its direction is nearly 'vertical'. Let us continue the trajectory through an angular domain, maintaining its 'verticality'. Denote the point of its intersection with the plane $t = t_\alpha$ by x^- and the point of intersection of the same plane with a true (broken) integral curve by x^+, then, to within infinitesimals of higher order, we will obtain from the right-angled triangle the discontinuity conditions (2.5).

Indeed, let us replace the equation of the discontinuity surface $P_\alpha = 0$ by the equation of the tangential plane

$$\left(\frac{\partial P_\alpha}{\partial x}\right)'_{M_\alpha} x + \left(\frac{\partial P_\alpha}{\partial t}\right)^-_{M_\alpha} (t-t_\alpha) = 0.$$

From (2.11), it follows that

$$\left(\frac{\partial P_\alpha}{\partial x}\right)'_{M_\alpha} = \left(\frac{\partial F_\alpha}{\partial z}\right)_{M_\alpha}, \qquad \left(\frac{\partial P_\alpha}{\partial t}\right)^-_{M_\alpha} = \left(\frac{dF_\alpha}{dt}\right)^-_{M_\alpha}.$$

By substituting these derivatives into the equation obtained for the tangent plane, and solving it for $t-t_\alpha$, we get

$$t_\alpha - t = h^{-\prime}x.$$

Ignoring infinitesimals of second order, the inclination of the trajectory in the angular domain is ξ. Thus

$$x^+ - x^- = \xi(t_\alpha - t_1),$$

where t_1 is the value of t at the entrance of trajectory into the angular domain, i.e. at $x = x^-$. Hence

$$x^+ - x^- = \xi h^{-\prime} x^- = B^- x^-,$$

thus proving the point.

3. The Liapounoff transformation

Apply a Liapounoff transformation

$$x = L(t)y, \tag{3.1}$$

where $$L(t) = X(t)e^{-At}, \qquad A = \frac{1}{\tau}\ln U, \tag{3.2}$$

to the linear approximation. Here L is the transformation matrix, X the fundamental matrix of the linear approximation (2.2)+(2.3), U the constant matrix in identity (1.6), y the column, composed of new variables $y_1, \ldots, y_n$.†

† Transformation (3.1) is called a Liapounoff transformation if all the elements of matrices L, L^{-1}, and dL/dt are bounded, when $t \geqslant t_0$.

Let Y denote the fundamental matrix of system (2.2)+(2.3), transformed to variables y_i. It then follows from (3.1) that $X(t) = L(t)Y(t)$, or, taking into account (3.2), we get $Y(t) = e^{At}$.

It follows directly from this that the equation of linear approximation in variables y_i becomes

$$\frac{dy}{dt} = Ay \tag{3.3}$$

and the integral curves of the system of the linear approximation in the space (y, t) are continuous. Previously, the linear approximation in space

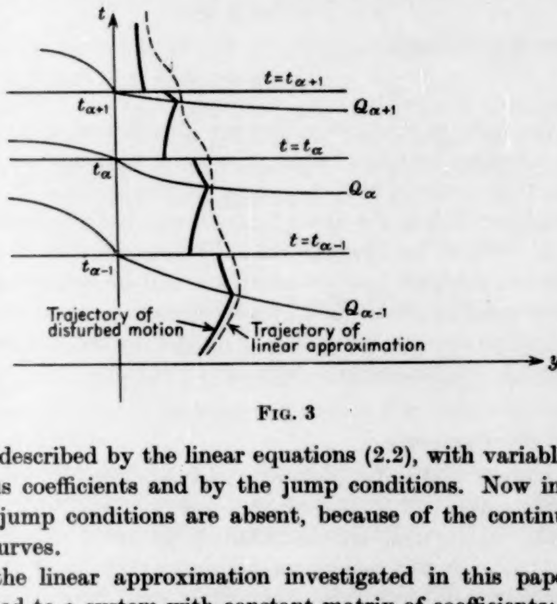

FIG. 3

(x, t) was described by the linear equations (2.2), with variable piecewise continuous coefficients and by the jump conditions. Now in the space (y, t) the jump conditions are absent, because of the continuity of the integral curves.

Thus, the linear approximation investigated in this paper may be transformed to a system with constant matrix of coefficients; but unlike the case dealt with by Liapounoff, the periodic matrix $L(t)$ is discontinuous at $t = t_\alpha$, since at these values of t, the columns of matrix $X(t)$ are discontinuous. Indeed, by (2.5) $X^+ = SX^-$, when $t = t_\alpha$, and hence

$$L^+ = SL^-. \tag{3.4}$$

Applying the Liapounoff transformation to the initial non-linear system (2.10), we obtain the system

$$\frac{dy}{dt} = q(y, t) \qquad \left(q = L^{-1}p - L^{-1}\frac{dL}{dt}y\right). \tag{3.5}$$

The discontinuity surfaces $P_\alpha(x, t) = 0$ are transformed into surfaces $Q_\alpha(y, t) = 0$, where $Q_\alpha(y, t) \equiv P_\alpha(L(t)y, t)$ (Fig. 3).

Unlike the surfaces $P_\alpha(x,t) = 0$, which are continuous and on the lines of intersection with the planes $t = t_\alpha$ having edges only, the surface $Q_\alpha(y,t) = 0$ has discontinuities along these lines of intersection, i.e. it is formed out of two continuous parts (at $t \leqslant t_\alpha$ and $t \geqslant t_\alpha$) intersecting the plane $t = t_\alpha$ along different curves, which pass through one point $t = t_\alpha$ of the t-axis (corresponding to point M_α). In the coordinates (x,t) the integral curves of the system (2.10) are continuous. The integral curves of (3.5), corresponding to these curves in space (y,t) are discontinuous at $t = t_\alpha$, since at these values of t the matrix $L(t)$ is discontinuous. At $t = t_\alpha$ we have

$$x = L^+y^+ = L^-y^-; \tag{3.6}$$

hence, on account of (3.4)

$$y^+ = (L^-)^{-1}S^{-1}L^-y^-. \tag{3.7}$$

Thus, in the space (y,t) the integral curve of undisturbed motion, as in space (x,t), coincides with the t-axis. The curve of disturbed motion in the space (y,t), as in space (x,t), has edges (on the surfaces $Q_\alpha = 0$), but in the space (y,t), unlike the space (x,t), it also has discontinuities on planes $t = t_\alpha$, defined by formula (3.7). The trajectories of the linear approximation in the space (y,t) are continuous and are defined by a system of linear differential equations (3.3) with coefficients constant everywhere. The space (y,t), as also the space (x,t), is divided by the surfaces $Q_\alpha = 0$ and the planes $t = t_\alpha$ into central and angular domains.

4. Proof of the theorems

Let the zero-solution of the linear approximation (1.3)+(1.4) be asymptotically stable. Then the solution $y = 0$ of system (3.3) is also asymptotically stable, and all characteristic roots of matrix A have negative real parts. In this case, as was shown by Liapounoff, there exists a positive-definite quadratic form $V(y)$ whose total derivative $(dV/dt)^0$, calculated by means of (3.3), is negative-definite:

$$V(y) > 0, \qquad \left(\frac{dV}{dt}\right)^0 < 0 \quad (y \neq 0).$$

Let us consider the variation of values of this function $V(y)$ along discontinuous integral curves of the non-linear system (3.5).

An integral curve passes through central and angular domains and has discontinuities on the plane $t = t_\alpha$. We will now show that the value of function V along trajectories in the central domain decreases and we will estimate the rate of decrease. In angular domains the value of V may increase. We will estimate the rate of increase and show that the increase

is compensated by jumps of function V on planes $t = t_\alpha$, adjoining angular domains.

In the central domain the linear approximation of non-linear system (3.5) is the system (3.3). Thus, in the central domain

$$q = Ay+(*). \tag{4.1}$$

Here and subsequently (*) and (**) stand for the sum of terms, the order of smallness of which, in regard to y is greater than one or two respectively. Then

$$\frac{1}{V}\left(\frac{dV}{dt}\right)^{00} = \frac{1}{V}\left(\frac{dV}{dt}\right)^{0} + \frac{1}{V}(**), \tag{4.2}$$

where derivatives $(dV/dt)^{00}$ and $(dV/dt)^{0}$ are calculated in accordance with systems (3.5) and (3.3) respectively.

The first term of the right-hand side of (4.2) is a continuous, homogeneous function of zero degree and it has a negative maximum $-\mu^2$. The second term tends to zero when $y \to 0$ and therefore, for any $\mu_1^2 < \mu^2$ and y sufficiently small, in consequence of (4.2),

$$\frac{1}{V}\left(\frac{dV}{dt}\right)^{00} \leqslant -\mu_1^2, \tag{4.3}$$

and thence

$$V \leqslant V^* e^{-\mu_1^2(t-t^*)}. \tag{4.4}$$

Here V^* and V denote the values of V at moments t^* and t ($t^* < t$) when the point of the integral curve is in the given central domain. Formula (4.4) defines the rate of decrease of function V along the integral curve in the central domain.

In the angular domain as $y \to 0$ (t then also tends to the respective t_α), the function q has finite limits $\eta^- = (L^-)^{-1}\xi$ and $\eta^+ = -(L^+)^{-1}\xi$ in the lower and upper angular domains, respectively. Thus, in an angular domain

$$\frac{1}{V^{\frac{1}{2}}}\frac{dV}{dt} = \frac{1}{V^{\frac{1}{2}}}\sum_k \frac{\partial V}{\partial y_k}\eta_k + \frac{1}{V^{\frac{1}{2}}}(*)$$

is bounded, when y is small enough, i.e. there is a K such that

$$\left|\frac{1}{V^{\frac{1}{2}}}\frac{dV}{dt}\right| < 2K;$$

hence

$$|V^{\frac{1}{2}}-(V')^{\frac{1}{2}}| < K|t-t'|,$$

where V and V' are the values of V at t and t' in one and the same angular domain. Since $|t-t'|$ is a quantity at least as small as the first order of magnitude in respect to y, the ratio V/V' is bounded when y is small enough, i.e. there exists a number $N > 1$, such that

$$V \leqslant NV', \tag{4.5}$$

i.e. the coefficient of increase of V in an angular domain is finite. We will now show that the variation of V in an angular domain is compensated by a corresponding jump $V(y^+)-V(y^-)$. Assume, for instance, that the integral curve passes through the lower angular domain in spaces (x,t) and (y,t) from points x_1, t_1 and y_1, t_1 on the discontinuity surface to points (x,t_α) and (y^-,t_α) respectively on plane $t = t_\alpha$ (Fig. 4).

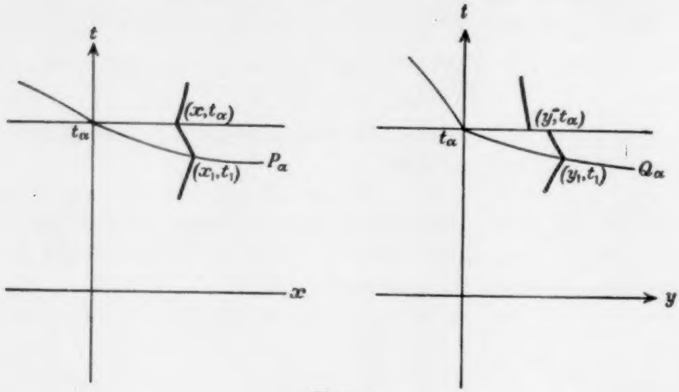

FIG. 4

Solve the equation of surface $P_\alpha(x,t) = 0$ with respect to t:†

$$t_\alpha - t = h^{-\prime}x + (*). \tag{4.6}$$

Since, in the lower angular domain $\lim p = \xi$, as $x \to 0$.

$$\begin{aligned} x - x_1 &= \int_{t_1}^{t_\alpha} p\,dt = \xi(t_\alpha - t_1) + (*) \\ &= \xi h^{-\prime}x_1 + (*) = B^- x_1 + (*), \end{aligned}$$

i.e.

$$x = Sx_1 + (*). \tag{4.7}$$

On the other hand,

$$x_1 = L(t_1)y_1 = L^- y_1 + [L(t_1) - L(t_\alpha - 0)]y_1 = L^- y_1 + (*). \tag{4.8}$$

From (3.4), (3.6), (4.7), and (4.8) we have

$$L^+y^+ = x = Sx_1 + (*) = SL^- y_1 + (*) = L^+ y_1 + (*). \tag{4.9}$$

By multiplying from the left by $(L^+)^{-1}$, we get

$$y^+ = y_1 + (*), \qquad V(y^+) = V(y_1) + (**) \tag{4.10}$$

† Since $x = Ly$, $y = L^{-1}x$, and the elements of the matrices $L(t)$ and $L^{-1}(t)$ are bounded, quantities infinitesimal in respect to y will be also infinitesimal in respect to x, and of the same order of magnitude, and vice versa. Hence we may use here the same designations (*) and (**).

and thus, when y is small enough,

$$e^{-\eta} < \frac{V(y^{+})}{V(y_1)} = 1 + \frac{1}{V(y_1)}(**) < e^{\eta}, \tag{4.11}$$

where η is a small positive quantity of the first order.

Let $T = \min_\alpha(t_{\alpha+1} - t_\alpha)$; let ν be an arbitrary positive number, smaller than μ, and let $\nu < \mu_1 < \mu$. Assume that $\eta < \frac{1}{2}(\mu_1^2 - \nu^2)T$. Let $\epsilon > 0$ be so small that inside the elliptic cylinder $V = \epsilon$ the inequalities (4.4), (4.5), and (4.11) hold good, and the interval of time Δt spent inside any angular domain contained in the cylinder $V = \epsilon$ is smaller than $\frac{1}{2}(\mu_1^2 - \nu^2)T/\mu_1^2$.

Such a choice of ϵ can be achieved, since it follows, from the periodicity and property (7) (section 1), that the number of central and angular domains under consideration is finite. Consider a sequence of moments of time $t_\alpha^* = \frac{1}{2}(t_\alpha + t_{\alpha+1})$. The corresponding planes $t = t_\alpha^*$ inside cylinder $V = \epsilon$ do not cross angular domains.

Let $\delta = \epsilon/N < \epsilon$. Take an initial point of an integral curve at $t = t_1^*$ inside the cylinder $V = \delta$. This point belongs to the central domain. When t varies over the interval $t_1^* \leqslant t \leqslant t_2^*$, function V at first decreases along an integral curve; then it may increase (the increase is compensated by a jump), and again decrease. Since the coefficient of increase does not exceed N, the integral curve on segment $t_1^* \leqslant t \leqslant t_2^*$ lies inside the cylinder $V = \epsilon$.

Now, assuming that $V_\alpha = V_{t=t_\alpha^*}$, we have

$$V_2 \leqslant V_1 e^{-\mu_1^2(T-\Delta t)+\eta} < V_1 e^{-\nu^2 T}. \tag{4.12}$$

Since, $V_2 < V_1 < \delta$, it follows that the point of an integral curve at $t = t_2^*$ will again be inside the cylinder $V = \delta$, and thence we can repeat the foregoing argument for the next interval $t_2^* \leqslant t \leqslant t_3^*$ of the integral curve, and so on. It follows, that any integral curve of the non-linear system, beginning at $t = t_1^*$ inside cylinder $V = \delta$, will always be inside cylinder $V = \epsilon$, and

$$V_\alpha \leqslant V_1 e^{-(\alpha-1)\nu^2 T}.$$

Thus, $\lim V_\alpha = 0$ when $\alpha \to \infty$. Since the inequality $V \leqslant NV_\alpha$ holds good in every interval $t_\alpha^* \leqslant t \leqslant t_{\alpha+1}^*$, $\lim V(t) = 0$ when $t \to \infty$. Thus, Theorem 1 is established.

Starting to prove Theorem 2, we assume that the absolute value of at least one of the roots of the characteristic equation of the linear approximation is greater than unity. Then one of the roots of the characteristic equation of matrix A has a positive real part. In this case there exists a quadratic form $V(y)$ which is positive for a certain value y_0 and, by (3.3),

has a positive-definite derivative

$$V_0 = V(y_0) > 0, \qquad \left(\frac{dV}{dt}\right)^0 > 0 \quad (y \neq 0).$$

Let
$$\kappa^2 = \min \frac{1}{V}\left(\frac{dV}{dt}\right)^0 \quad (V \geqslant V_0).$$

Then, in any central domain along an interval of an integral curve with $V \geqslant V_0$ we have

$$\frac{1}{V}\left(\frac{dV}{dt}\right)^{00} \geqslant \kappa_1^2, \tag{4.13}$$

when y is small enough.

Here $0 < \kappa_1 < \kappa$, and $(dV/dt)^{00}$ is a derivative, calculated by equation (3.5). Again, let $T = \min_\alpha(t_{\alpha+1}-t_\alpha)$ and $0 < \sigma < \kappa_1^2$.

Let $\epsilon > 0$ be small enough, so that inside the cylinder $V = \epsilon$ the inequalities (4.11) and (4.13) hold good, and let the time Δt spent in any angular domain contained in cylinder $V = \epsilon$ be smaller than

$$\frac{1}{2\kappa_1^2}(\kappa_1^2-\sigma^2)T.$$

Assume $\eta < \frac{1}{2}(\kappa_1^2-\sigma^2)T$. Then, instead of inequality (4.12) we have

$$V_2 \geqslant V_1 e^{+\kappa_1^2(T-\Delta t)-\eta} \geqslant V_1 e^{\sigma^2 T} \quad \{V_1 = V(y_0)\}.$$

This inequality will hold good, if at all values of the interval $t_1^* \leqslant t \leqslant t_2^*$, the integral curve lies inside the cylinder $V = \epsilon$. Then $V_2 \geqslant V_1 > 0$ and we may repeat our argument for the next interval of the integral curve, etc. At a certain moment of time the integral curve will leave the cylinder $V = \epsilon$, since the values of V_α increase quicker than the terms of a geometrical progression with a common factor $q = e^{\sigma^2 T} > 1$. As $V(y_0)$ can be made as small as we like, Theorem 2 is established.

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ON GOLDSTEIN'S THEORY OF LAMINAR SEPARATION

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[Received 31 October 1957]

SUMMARY

It is shown that the expansion assumed by Goldstein to describe the flow near separation in a laminar boundary layer is incomplete and that further terms which include powers of logarithms must be added. These terms are individually singular at separation. Although it cannot be inferred that the velocity profile must also be singular at separation, it is suggested that if the boundary layer is to continue downstream of separation the main stream must adjust itself so that these terms cannot appear. The solution may then be continued through separation by means of a power series into a region of reversed flow. However, it is shown that in addition to the power series an infinity of new terms may appear in the solution downstream of separation which is therefore no longer specified uniquely by the mainstream velocity and the velocity profile at the beginning of the boundary layer.

1. Résumé

TEN years ago Goldstein (1) published an important paper on the flow in a two-dimensional boundary layer near separation, i.e. near the point where the skin friction vanishes. He considered the non-dimensional form of the boundary layer equations

$$-u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -U\frac{dU}{dx}+\frac{\partial^2 u}{\partial y^2}, \qquad u = \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x} \tag{1}$$

in which x measures distance *upstream* along the wall from the point of separation O and y distance normal to the wall as shown in Fig. 1. Further, u, v are the velocity components in the directions of x *decreasing* and y increasing respectively and ψ is the stream function. The boundary conditions are that $u = v = 0$ at $y = 0$ and that u tends to prescribed functions $U(x)$, $k(y)$ as $y \to \infty$ and as x tends to some positive value respectively. The non-dimensional form chosen was such that

$$U(0) = U'(0) = 1$$

and therefore near $x = 0$ we may write

$$U\frac{dU}{dx} = 1+\sum_1^\infty P_r x^r, \tag{2}$$

where P_r are prescribed constants.

Goldstein assumed that at $x = 0$, where the skin friction vanishes and the boundary layer separates,

$$u = \tfrac{1}{2}y^2 + \sum_{3}^{\infty} a_r y^r, \tag{3}$$

where the a_r are unknown to begin with. The coefficient of y is zero in (3) because $(\partial u/\partial y)_{y=0} = 0$ at separation and the coefficient of y^2 is $\frac{1}{2}$ as a

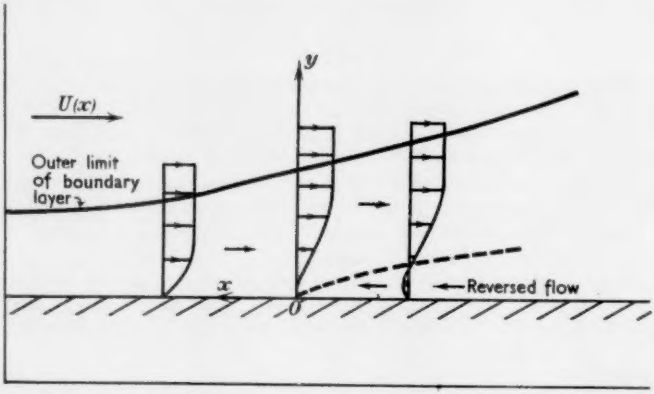

FIG. 1. Velocity profiles in the boundary layer near separation

result of the choice of units. His object was to determine the behaviour of ψ in the neighbourhood of $x = 0$ by solving (1) with boundary conditions $u = v = 0$ at $y = 0$, and (3). Although the a_r are unknown initially, if his procedure were correct, they would all be determined in terms of the P's and one other parameter. His method was to write

$$\xi = x^{\frac{1}{4}}, \qquad \eta = y/(2^{\frac{1}{4}}x^{\frac{1}{4}}), \qquad \psi = 2^{\frac{3}{4}}\xi^3 \sum_{r=0}^{\infty} \xi^r f_r(\eta), \tag{4}$$

substitute into (1) and equate powers of ξ. The boundary conditions on f_r are that

$$f_r(0) = f_r'(0) = 0 \quad \text{and} \quad \lim_{\eta\to\infty} \frac{f_r'}{\eta^{r+2}} = 2^{\frac{1}{4}r} a_{r+2}. \tag{5}$$

As a result of the solution he found that

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 2^{\frac{3}{4}} \sum_{r=1}^{\infty} x^{\frac{1}{4}(r+1)} \alpha_r \tag{6}$$

where $\alpha_r = \frac{1}{2}f_r''(0)$.

On substituting (4) into (1) and comparing coefficients of ξ it was found that

$$f_0(\eta) = \tfrac{1}{6}\eta^3, \qquad f_1 = \alpha_1 \eta^2, \tag{7}$$

and f_r satisfied

$$f_r''' - \tfrac{1}{2}\eta^3 f_r'' + \tfrac{1}{2}(r+4)\eta^2 f_r' - (r+3)\eta f_r = G_r(\eta) \quad (r \geqslant 2). \tag{8}$$

The right-hand side G_r of (8) is a known function of f_s when $s < r$ and, if $\frac{1}{4}r$ is an integer, of the pressure coefficient $P_{\frac{1}{4}r}$; it has the form

$$-2(r+2)\alpha_1 \alpha_{r-1} \eta^2 + \bar{G}_r(\eta), \tag{9}$$

where $\bar{G}_r$ is explicitly independent of α_{r-1} and has a double zero at the origin.

Since one of the complementary functions of (8) is η^2 it is clear that α_r cannot be determined so long as the expansion of ψ is not continued beyond f_r. Goldstein (**1**), however, suggested a means by which they could all be expressed in terms of the P's and α_1, and with Jones (**2**) evaluated α_2, α_3, and α_4 as follows. The other two complementary functions of (8) are g_r, h_r of which g_r has a simple zero at $\eta = 0$, $h_r(0) \neq 0$. Both g_r, h_r may be exponentially large at infinity but a suitable combinaton of them can always be found which is algebraic at infinity. The appropriate solution of (8) is

$$f_r = 4\alpha_1 \alpha_{r-1}(\eta - g_r) + j_r(\eta),$$

where j_r satisfies (8) with G_r replaced by $\bar{G}_r$ and hence is independent of α_{r-1}. Now if we choose j_r to have a double zero at $\eta = 0$ it will in general have an exponentially large component at infinity which can be cancelled by a suitable multiple of g_r provided only that g_r is also exponentially large, i.e. $r \neq 4n+2$, where n is an integer. Hence if the solution is known as far as f_{r-1} in terms of α_1 and the P's, (8) will determine α_{r-1} except if $r = 4n+2$. In the latter case however g_r is a polynomial and instead $\bar{G}_r$ must satisfy the condition

$$\int_0^\infty \bar{G}_r e^{-\frac{1}{4}\eta^4}(2\eta g_r - \eta^2 g_r')\, d\eta = 0. \tag{10}$$

The first difficulty occurs when $r = 6$ and Jones (**2**) showed that in this case the value of the integral in (10) is $-4\alpha_1^6 \pm 4\alpha_1^6$ so that it is distinctly possible for (10) to hold when $r = 6$. Goldstein pointed out that in this case the way is clear for a complete determination of ψ, for α_5 would then quite possibly be determined from the integral condition at $r = 10$ and so on. Thus all the α_r could be found in terms of α_1 and the P's, and presumably α_1 could then be found from the condition that $u \to U$ as $y \to \infty$. Since f_r would be completely determinate all the a_r would be

known and all separating boundary layers with the same main stream would have the same separation profile.

2. Discussion

Among the many aspects of this important work there are two of special interest here. First, does the condition (10) really hold when $r = 6$? Jones's work indicates that it may possibly not hold and further it is unusual to find a mathematical theory of this kind hanging by a thread. Moreover in his corresponding study of laminar wakes, Goldstein (**3**) encountered a similar integral to (10) which he showed to be non-vanishing.

Second, if α_1 determines the asymptotic behaviour of ψ near separation why is it that when $\alpha_1 = 0$ an infinity of arbitrary constants appear in the solution (**1**, section 6), α_{4n+3} then being arbitrary?

These questions may be answered by making use of a theory of asymptotic expansions given in a previous paper (**4**). There it is shown that certain asymptotic expansions, developed in connexion with problems in boundary layer theory, are incomplete and it is also shown how the additional terms necessary to complete them may be calculated. Associated with the new terms, however, are arbitrary constants, which cannot be determined by the method and depend in some way on the initial conditions.

Before doing this however it is necessary to change the point of view slightly. Goldstein's plan was to assume that u could be expanded as a power series in y at $x = 0$, the point of separation, and deduce ψ as a power series in $\xi = x^{\frac{1}{4}}$, whose coefficients were functions of $\eta = y/2^{\frac{1}{2}}x^{\frac{1}{4}}$, which would be valid in $x \geqslant 0$, $y \geqslant 0$. Unfortunately, the modification to u, which is necessary, contains terms which tend to infinity as $x \to 0+$ for fixed $y > 0$. The initial assumption that u can be expanded as a power series in y when $x = 0$ is therefore invalid.

The expansion for ψ obtained is however not necessarily invalid in the entire neighbourhood of the origin. For Goldstein could equally well have assumed, instead of (3), the expansion of $(\partial u/\partial y)_{y=0}$ as the series of ascending powers of $x^{\frac{1}{4}}$ which is given in (6), deducing from it and the other boundary conditions the behaviour of ψ in the neighbourhood of the origin. Looked at from this point of view the aim would then be to determine if possible the behaviour of u when $x = 0$ and $y \neq 0$. The modification to his solution adds more terms to his expansion for $(\partial u/\partial y)_{y=0}$ without impairing the validity of the basic notion. Since some of the new terms tend to infinity as $x \to 0+$ for fixed $y > 0$, however, it is no longer

possible to extend the range of validity of the expansion for ψ as far as this line. It is only valid in fact when

$$x \geqslant 0, \qquad y \leqslant Ax^{\frac{1}{4}},$$

where A is any finite positive number.

The problem may be posed in the following way. The simplest expansion of ψ about the origin is a regular double power series in x and y. According to it $(\partial u/\partial y)_{y=0}$ may be expanded as a series of integral powers of x near $x = 0$. However, in the only numerical solution of the boundary layer equations in the vicinity of separation which is available, $(\partial u/\partial y)_{y=0}$ is not of this form. How can the expansion of ψ be generalized to account for the computed behaviour of $(\partial u/\partial y)_{y=0}$? The correct way is to assume that ψ can be expanded as a series of powers of ξ and $\log \xi$ whose coefficients are functions of η which can be determined, apart from certain numerical factors, by substitution in the differential equation (1).

3. The modification to the theory

We shall assume as a new starting-point therefore that

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 2^{\frac{1}{2}} \sum_{r=1}^{\infty} x^{\frac{1}{4}(r+1)}\alpha_r \tag{6}$$

and the appropriate boundary conditions at $\eta = 0$ are

$$f_r(0) = f_r'(0) = 0, \qquad f_r''(0) = \tfrac{1}{2}\alpha_r.$$

In order that the motion at $x = 0+$, $y \neq 0$ should not be violent we must require in addition that f_r be not exponentially large at infinity. The α_r are not now all independent and it follows that if (6) is the correct expansion for $(\partial u/\partial y)_{y=0}$ then the new conditions are equivalent to (5). The solution obtained with the new standpoint is thus equivalent to the other and leads to the same difficulties. We now ask whether (6) is the most general form for $(\partial u/\partial y)_{y=0}$; can more terms be added to (6) without impairing the validity of the corresponding expansion of ψ? Using the results in (4) these questions may be answered and further the theory rendered self-consistent. These imply that an infinite number of infinite series of ascending powers of ξ whose coefficients are the products of a power of $\log \xi$ and a function of η can be added to (4) and corresponding series to (6). Each series is associated with one of the integral conditions (10) which occurred in Goldstein's original theory in a way which is exemplified by the case $r = 6$. Knowing that the new terms may have logarithmic factors it is found by trial that the most general form of ψ possible is

$$\psi = 2^{\frac{3}{2}}\xi^3 \sum_0^6 \xi^r f_r(\eta) + 2^{\frac{3}{2}}\xi^8 \log \xi [F_5(\eta) + \xi F_6(\eta)] + O(\xi^{10} \log \xi), \tag{11}$$

whence

$$u = 2\xi^2 \sum_0^6 \xi^r f'_r(\eta) + 2\xi^7 \log\xi[F'_5(\eta) + \xi F'_6(\eta)] + O(\xi^9 \log\xi)$$

and

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 2^{\frac{1}{2}}\xi \sum_0^6 \xi^r f''_r(0) + 2^{\frac{1}{2}}\xi^6 \log\xi[F''_5(0) + \xi F''_6(0)] + O(\xi^8 \log\xi).$$

The reason is as follows. First let the term of lowest order which can be added to (4) be $\xi^{3+s}F_s(\eta)$ where $s > 0$. Then $F_s(\eta) = \alpha_s \eta^2$, where α_s is a constant. This new term initiates a series of terms just as $2^{\frac{1}{2}}\alpha_1 \xi^4 \eta^2$ does in the original series. The equation for the next term in the new series, which is proportional to ξ^{3+2s} or ξ^{4+s} according as $s < 1$ or $s > 1$, is of the form (8) but the right-hand side is proportional to η^2. Hence $\bar{G} = 0$ and the term is exponentially large at $\eta = \infty$ except if $\alpha_s = 0$ or if $s = 4r+1$, r integer, when it is a polynomial, and $\xi^{3+s}F_s(\eta)$ is already in (4). Thus no extra terms proportional to powers of ξ can be added to (4). Next let the term of the lowest order which can be added to (4) be $\xi^{3+t}(\log\xi)^{t_1}F_t(\eta)$ where $t > 0$. As before $F_t(\eta) = \alpha_t \eta^2$ where α_t is a constant and $t = 4r+1$, r integer. The constant α_t is not now determined by the equation for the term proportional to $\xi^{4+t}(\log\xi)^{t_1}$ but by the equation for the term proportional to $\xi^{4+t}(\log\xi)^{t_1-1}$, which leads to an integral condition like (10). However, the $\bar{G}$ is zero, except if $t_1 = 1$, $r = 1, 2,\ldots$, when it depends on $f_s(\eta)$, $s \leqslant 4r+1$. Thus α_t need not be zero if $t = 4r+1$, $t_1 = 1$, accounting for the terms in (11).

Substituting (11) into (1) we find that f_r, $r < 6$, satisfies the same equation as before. The equation for F_5 is

$$F'''_5 - \tfrac{1}{2}\eta^3 F''_5 + \tfrac{9}{2}\eta^2 F'_5 - 8\eta F_5 = 0, \tag{12}$$

of which the appropriate solution is $F_5 = \beta_5 \eta^2$ where β_5 is a constant. The equation for f_5 is unaffected by F_5 and the equation for F_6 is

$$F'''_6 - \tfrac{1}{2}\eta^3 F''_6 + 5\eta^2 F'_6 - 9\eta F_6 = -16\alpha_1 \beta_5 \eta^2, \tag{13}$$

with solution

$$F_6 = 4\alpha_1 \beta_5(\eta - g_6) + \beta_6 \eta^2 = -\tfrac{4}{15}\alpha_1 \beta_5\left(\eta^5 - \frac{\eta^9}{84}\right) + \beta_6 \eta^2. \tag{14}$$

The equation for f_6 is now

$$\begin{aligned} f'''_6 - \tfrac{1}{2}\eta^3 f''_6 + 5\eta^2 f'_6 - 9\eta f_6 &= -16\alpha_1 \alpha_5 \eta^2 + \bar{G}_6 + F_6 f''_0 - F'_6 f'_0 + F_5 f''_1 - F'_5 f'_1 \\ &= -2\alpha_1 \beta_5(\eta^2 - \tfrac{1}{5}\eta^6 + \tfrac{1}{180}\eta^{10}) - 16\alpha_1 \alpha_5 \eta^2 + \bar{G}_6 \end{aligned} \tag{15}$$

and the solution of (15) with a double zero at the origin is not exponentially large at infinity if

$$\int_0^\infty \left\{\bar{G}_6 - 2\alpha_1 \beta_5\left(\eta^2 - \frac{\eta^6}{5} + \frac{\eta^{10}}{180}\right)\right\}\left\{\eta^2 - \frac{\eta^6}{5} + \frac{\eta^{10}}{180}\right\} e^{-\frac{1}{4}\eta^4}\, d\eta = 0$$

from (10). Using Jones's value for the integral in (10) when $r = 6$ we have

$$\tfrac{128}{45}\,2^{\frac{1}{4}}(\tfrac{1}{4})!\,\alpha_1\beta_5 = -4\alpha_1^6,$$

whence

$$\beta_5 \doteqdot -\alpha_1^5. \tag{16}$$

In principle we can now proceed as far as we like with the asymptotic expansion. Extra terms containing $\log\xi$ must be added at the 7th and 8th stages; the corresponding β's may be determined in terms of α_1, while α_7, α_8 depend on α_1 and α_5 which now emerges as a new arbitrary constant. At the 10th stage however new forcing terms containing $(\log\xi)^2$ appear, and to cancel these new terms containing $(\log\xi)^3$ must be introduced at the 9th stage of the asymptotic expansion of ψ. As a result of satisfying the three integral conditions associated with the 10th stage, all the functions which have appeared up to this stage are determinate in terms of *three* arbitrary constants α_1, α_5, and α_9. The development may be continued indefinitely and finally the asymptotic expansion of ψ appears as an infinite double series in powers of ξ and $\log\xi$ whose coefficients are functions of η and contain an infinite number of arbitrary constants.

One interesting feature of these additional terms is that u will have a logarithmic singularity at $x = 0$. Thus when x is small the additional terms in (11) are dominated by

$$\frac{\alpha_1\beta_5}{10080}y^9\log x, \tag{17}$$

which tends to infinity as $x \to 0$. It does not necessarily follow that because certain terms, in the expansion of u assumed in this paper, tend to infinity as $x \to 0$ for $y \neq 0$, u itself is infinite on the line $x = 0$. As a counter-example consider $Z = y^2x^y$. Then as $x \to 0+$, $Z \to 0$ for all $y \geqslant 0$. On the other hand, for any $x \neq 0$ we may expand Z as a series of powers of y, viz.

$$Z = \sum_{n=0}^{\infty}\frac{y^{n+2}(\log x)^n}{n!},$$

every term of which tends to infinity as $x \to 0+$. It would be exceedingly difficult, if not impossible, to decide whether u has a logarithmic singularity at $x = 0$ by numerical integration from some initial profile. For example, Leigh (5) has given a numerical solution correct to six places of decimals for a certain boundary layer to within, in his notation, a distance 0·00014 of separation. For his values of $x = 0{\cdot}00014$ and of $y = 2$, which is just outside the region in which a comparison with Goldstein's asymptotic expansion is possible, the value of (17) above is only 10^{-5}. Further, to increase (17) by a factor of ten at the same value of y, we should have to be within 10^{-8} of $x = 0$. The presence of the logarithmic

terms makes it clear why Goldstein's original method of using the separation profile as one boundary condition has to be abandoned, for the singularities to which they lead make nonsense of the assumption implicit in (3).

The present modification to Goldstein's theory enables us to resolve the difficulties mentioned earlier in this paper. First, it does not matter whether the integral condition (10) with $r = 6$ is satisfied or not. If not then logarithmic terms must be introduced at the 5th and subsequent stages. If it is satisfied then their appearance is deferred to the 9th stage. Second, there is always an infinite sequence of arbitrary constants in the asymptotic expansion, which are α_{4n+1} if $\alpha_1 \neq 0$ and α_{4n+3} if $\alpha_1 = 0$. Finally it has been shown (**4**) that in the related problem of heat conduction in one dimension a similar set of arbitrary constants appear in a corresponding asymptotic expansion which can be expressed in terms of the initial temperature profile. Here therefore it is expected that the unknown α's depend in some way on the initial profile $k(y)$ of u.

4. The solution downstream of separation ($x < 0$)

Although it is not clear whether the extra logarithmic terms make u singular on the line $x = 0$, they render difficult any extension of the solution downstream of it. In particular the method suggested by the author (**6**) for a liquid which breaks away from the wall at $x = 0$ and is thereafter bounded on one side by a free streamline, cannot directly handle these new terms. If the mainstream does not break away† from the wall at $x = 0$, which it need not do, the only way in which the boundary layer could continue downstream of $x = 0$ seems to be if $\alpha_1 = 0$. This implies that the mainstream must satisfy some condition at $x = 0$ and cannot therefore be specified completely independently of the boundary layer.

If $\alpha_1 = 0$, a formal solution for ψ valid both upstream and downstream of $x = 0$ may be obtained as a double power series in x and y, as Goldstein (**1**) has pointed out. This cannot be the whole story, however, since it implies that the solution in $x < 0$ is completely specified by $U(x)$, the mainstream velocity, and by the solution in $x > 0$. However, from the

† The identification in the literature of the point of separation x_s with the point x_0 at which the skin friction vanishes may lead to some confusion. In this paper x_s is the earliest point at which the fluid moving forward *in the boundary layer* is no longer adjacent to the wall, but is separated from it by a layer of fluid moving in the reverse direction. It coincides therefore with x_0. However, the word separation is sometimes used to describe the point x_B at which the main stream is observed to leave the neighbourhood of the wall. In this paper to distinguish the two phenomena we use the term breakaway to describe x_B. There is no satisfactory theory at present which can predict where breakaway occurs. It may be observed to occur even if the skin friction does not vanish and conversely the mainstream has remained near to the wall even when the boundary layer has separated. If breakaway and separation both occur then x_B is usually just downstream of $x_s = x_0$.

form of the boundary layer equations it follows that disturbances are propagated by convection, with velocity u, in the x direction and by diffusion in the y direction. Thus upstream of separation $\psi(x_i, y_i)$ can only be modified by a disturbance at x, y if $x \geqslant x_1$. Downstream of separation the boundary layer can be divided into two parts in one of which $u > 0$ and in the other $u < 0$. Hence wherever in $x < 0$ the disturbance is introduced it diffuses into both parts of the boundary layer and then spreads by convection into the whole boundary layer downstream of separation. The boundary layer upstream of separation is of course unaffected except indirectly through any consequent modification of the mainstream.

Mathematically this means that in $x < 0$ extra terms, additional to the double power series, must appear in the solution, which are not fully determinate by the conditions in the neighbourhood of $x = 0$. A method is given here by which these terms may be found, and it will appear that at $x = 0$ not only do they vanish but so also do all their derivatives with respect to x; thus the join between the flows upstream and downstream of separation is perfectly smooth.

Let the stream function obtained by the expansion in a double power series be ψ_0 equal to

$$\tfrac{1}{6}y^3+\tfrac{1}{2}\alpha_3 xy^2+\bar{\psi}_0 \tag{18}$$

near $x = y = 0$ where α_3 is a positive constant and $\bar{\psi}_0$ is $O\{y^2(x^2+y^2)\}$. Further let the complete stream function $\psi = \psi_0+\psi_1$ where $\psi_1 \ll \psi_0$ when x is small. Near $x = 0$, therefore, ψ_1^2 may be neglected in comparison with ψ_1, which, on substituting into (1), satisfies

$$\frac{\partial^3\psi_1}{\partial y^3}+\frac{\partial\psi_0}{\partial y}\frac{\partial^2\psi_1}{\partial x\partial y}-\frac{\partial^2\psi_0}{\partial y^2}\frac{\partial\psi_1}{\partial x}+\frac{\partial\psi_1}{\partial y}\frac{\partial^2\psi_0}{\partial x\partial y}-\frac{\partial^2\psi_1}{\partial y^2}\frac{\partial\psi_0}{\partial x} = 0 \tag{19}$$

with boundary conditions

$$\psi_1 = \frac{\partial\psi_1}{\partial y} = 0 \text{ when } y = 0,\ x \leqslant 0, \qquad \psi_1 = 0 \text{ when } x = 0,\ y \geqslant 0.$$

We look for a solution of the form

$$\psi_1 = \Psi_1(\theta, x), \quad \text{where } \theta = -y/\alpha_3 x, \tag{20}$$

assuming that while $\partial\Psi_1/\partial x$ may be much larger than Ψ_1, $\partial\Psi_1/\partial\theta$ is of the same order as Ψ_1. This assumption may be verified *a posteriori* because Ψ_1 has an essential singularity at $x = 0$. The equation satisfied by Ψ_1 is

$$\frac{\partial^3\Psi_1}{\partial\theta^3}+\tfrac{1}{2}\alpha_3 x[\theta(\theta-2)+O(x)]\frac{\partial^2\Psi_1}{\partial x\partial\theta}-\alpha_3 x[(\theta-1)+O(x)]\frac{\partial\Psi_1}{\partial x} = O(x^3\Psi_1) \tag{21}$$

where the terms $O(x)$ are of the form x multiplied by a double power series

in x, θ and the terms $O(x^3\Psi_1)$ are of the form x^3 multiplied by a double power series in x, θ and either Ψ_1, $\partial\Psi_1/\partial\theta$, or $\partial^2\Psi_1/\partial\theta^2$. The simplest appropriate form for Ψ_1, but not necessarily the only one, is

$$\Psi_1 = \left|\exp\left\{\frac{\beta}{3\alpha_3 x^3}+\frac{\gamma}{2\alpha_3 x^2}+\frac{\delta}{\alpha_3 x}\right\}\right| \sum_{r=0}^{\infty} (-x)^{r+n} F_r(\theta) \tag{22}$$

where β, γ, δ, n are constants of which β must be positive and $F_r(\theta)$ is a function of θ only satisfying the boundary conditions

$$F_r(0) = F_r'(0) = 0 \quad \text{and} \quad F_r(\theta) \not\to \infty \text{ exponentially as } \theta \to \infty. \tag{23}$$

The last condition is necessary because Ψ_1 must be bounded as $y \to \infty$ and because the disturbances which affect Ψ_1 are only propagated in the direction of x increasing near the plate and reach large y only through diffusion. The equation satisfied by F_0 is

$$F_0''' - \tfrac{1}{2}\beta\theta(\theta-2)F_0' + \beta(\theta-1)F_0 = 0, \tag{24}$$

which is too complicated to lead to a simple solution. Here therefore it will only be shown that solutions exist for an infinite number of acceptable values of β and that then (22) contains three arbitrary constants. Differentiating (24) with respect to θ gives

$$F_0^{\text{iv}} - \tfrac{1}{2}\beta\theta(\theta-2)F_0'' + \beta F_0 = 0.$$

We now suppose that β is large so that βF_0 may be neglected in comparison with F_0^{iv}. Thus the problem is reduced to solving

$$F_0^{\text{iv}} - \tfrac{1}{2}\beta\theta(\theta-2)F_0'' = 0$$

subject to the boundary conditions $F_0'''(0) = 0$, $F_0'' \not\to \infty$ exponentially as $\theta \to \infty$. The range of θ has to be divided into four parts and each considered separately:

(*a*) $\theta > 2$ and $\beta^{\frac{1}{3}}(\theta-2)$ large. Here the appropriate solution is

$$F_0'' \sim [\theta(\theta-2)]^{-\frac{1}{4}} \exp\left\{-\beta^{\frac{1}{2}} \int_2^\theta [\theta(\theta-2)]^{\frac{1}{2}}\, d\theta\right\},$$

where $\sim$ means that the ratio of the left- and right-hand sides tends to a constant value as $\beta \to \infty$. Then near $\theta = 2$

$$F_0'' \sim (\theta-2)^{-\frac{1}{4}} \exp\{-(2\beta)^{\frac{1}{2}}(\theta-2)^{\frac{3}{2}}\}. \tag{25}$$

(*b*) $\beta^{\frac{1}{3}}(\theta-2) = O(1)$. Here to match (25)

$$F_0'' \sim \beta^{\frac{1}{6}}(\theta-2)^{\frac{1}{2}} K_{\frac{1}{3}}\{\tfrac{2}{3}(2\beta)^{\frac{1}{2}}(\theta-2)^{\frac{3}{2}}\}$$

and when $\beta^{\frac{1}{3}}(\theta-2)$ is large and negative

$$F_0'' \sim (2-\theta)^{-\frac{1}{4}} \sin\{\tfrac{1}{4}\pi - \tfrac{2}{3}(2\beta)^{\frac{1}{2}}(2-\theta)^{\frac{3}{2}}\}. \tag{26}$$

(c) $0 < \theta < 2$; $\beta^{1/2}\theta$, $(2-\theta)\beta^{1/2}$ large and positive. Here, to match (26),

$$F_0'' \sim [\theta(2-\theta)]^{-1/4} \sin\left\{\beta^2 \int\limits_0^\theta [\theta(2-\theta)]^{1/2}\, d\theta - \tfrac{1}{2}\pi\beta^{1/2} + \tfrac{1}{4}\pi\right\}$$

and near $\theta = 0$ $\quad F_0'' \sim \theta^{-1/4} \sin\{\tfrac{1}{4}\pi - \tfrac{1}{2}\pi\beta^{1/2} + \tfrac{2}{3}(2\beta)^{1/2}\theta^{3/2}\}$. (27)

(d) $\theta\beta^{1/2} = O(1)$. Here since $F_0'''(0) = 0$

$$F_0'' \sim \beta^{1/2}\theta^{1/2} Y_{1/3}\{\tfrac{2}{3}(2\beta)^{1/2}\theta^{3/2}\},$$

and when $\theta\beta^{1/2}$ is large

$$F_0'' \sim \theta^{-1/4} \sin\left\{\tfrac{2}{3}(2\beta)^{1/2}\theta^{3/2} - \frac{5\pi}{12}\right\}. \tag{28}$$

Hence, to match (27) and (28),

$$2s\pi - \frac{5\pi}{12} = \tfrac{1}{4}\pi - \tfrac{1}{2}\pi\beta^{1/2},$$

where s is a large positive integer. Thus

$$\beta = \tfrac{16}{9}(3s-1)^2 \tag{29}$$

and there are an infinite number of suitable values of β.

Each such β generates an independent solution of (21), of the form (22), containing three arbitrary constants, which may be taken to be $F_0''(0)$, $F_1''(0)$, $F_2''(0)$. Given these in addition to β all other terms in (22) may be found. First γ is found from the equation for $F_1(0)$, which is

$$F_1''' - \tfrac{1}{2}\beta\theta(\theta-2)F_1' + \beta(\theta-1)F_1 = \gamma F_0''' + G_1(\theta), \tag{30}$$

where G_1 is a linear function of F_0 but is independent of γ. Since the solution of the homogeneous equation for F_1 which satisfies the conditions at $\theta = 0$ also satisfies the condition at $\theta = \infty$, a solution of (30) satisfying these conditions is only possible when

$$\int\limits_0^\infty F_0'''(\gamma F_0''' + G_1)\, d\theta = 0, \tag{31}$$

which determines γ in terms of β only. Since an arbitrary multiple of F_0 may be added to F_1, $F_1''(0)$ is arbitrary. From the equation for F_2, δ may be found in terms of $F_0''(0)$, $F_1''(0)$, while $F_2''(0)$ is arbitrary. Similarly n is determined from the equation for F_3 and $F_r''(0)$ ($r \geqslant 3$) from the equation for F_{r+1}, thus formally completing the solution.

The form of Ψ_1 obtained above is very acceptable, for it has an essential singularity at $x = 0$ which ensures that it, and all its derivatives with respect to x, tend to 0 as $x \to 0-$, so that the join with the solution in $x > 0$ is perfectly smooth. Further, as would be expected, because there are an infinite number of suitable Ψ_1 containing in all an infinite number

of arbitrary constants, the solution in $x < 0$ is capable of an extremely wide variation depending on the boundary conditions beyond separation.

The situation is similar to that in the theory of the impulsive motion of a flat plate in an incompressible fluid studied by the author (**7**). There, if x denotes distance from the leading edge of the plate, t time from the start of the motion, and U the constant velocity of the plate, a similar difficulty arises in the region $0 < \xi < 1$ where $\xi = x/Ut$. The stream function ψ is known at $\xi = 0$ and at $\xi = 1$ but a step by step integration cannot be started from either point because the flow depends on boundary conditions at $\xi = 0$ and at $\xi = 1$. Essential singularities develop at $\xi = 0$, 1 containing an infinite number of unknown constants which depend in some way on the conditions at $\xi = 0$, 1. There is a close similarity between the essential singularities at $\xi = 0$ and n the present problem, the forms being alike and both being centred on $y = 0$.

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SIMILARITY FLOWS BEHIND STRONG SHOCK WAVES

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[Received 3 October 1957]

SUMMARY

Similarity solutions describing the flow of a perfect gas behind strong shock waves are investigated for the three cases of plane, cylindrical, and spherical symmetry. The flow is caused by an expanding piston, and the total energy increases as a power of the time. The ratio of kinetic to internal heat energy of the gas is computed and it is found that in the case of a piston which is expanding at a uniform rate there is equipartition of energy.

1. Introduction

In this paper a class of exact solutions of the equations governing the motion of a perfect gas is given and the properties of the solutions discussed. The flows depend on one spatial coordinate only. While certain of these solutions have been studied at different times (**1**, **2**, **3**, **4**), it is nevertheless of interest to examine a general set of such solutions in order to gain a better understanding of similarity flows; in particular the problem of the occurrence of singularities in the flow and the fitting of appropriate boundary conditions is investigated in some detail. The solutions to be discussed are of the progressing wave type, or similarity solutions. These are obtained by assuming the solutions to be of a certain form, which is given below, and by virtue of this assumption reducing the mathematical problem to one involving ordinary, instead of partial differential equations. This simplification is achieved by postulating that the dependent variables in the problem—the pressure, density, and velocity—can each be written as the product of a function of the time, and a function of a single variable x which is of the type

$$x = rt^{-\lambda}, \tag{1.1}$$

where r is the distance from the origin, t is the time, and λ is a constant which is determined, effectively, from considerations of energy. Such a set of solutions can then be employed to describe the flow behind an infinitely strong shock which is advancing into a gas at rest. It is possible moreover, (**4**), to treat the case in which the density distribution ahead of the shock front has the form

$$\rho = \rho_c r^{-\alpha}, \tag{1.2}$$

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where α is a constant. When the density of the undisturbed gas is uniform, and it is assumed that the total energy carried by the flow remains constant, the resulting solution, first obtained by Sir Geoffrey Taylor (**2**), describes the initial stages of an intense explosion. Now the total energy in the flow is the sum of the kinetic and internal heat energies of the gas; in practice there will be losses due to dissipative effects while there will also be a gain in the internal heat energy as the shock front advances and encloses more of the quiescent gas. However, this increase has effectively been omitted because of the assumption that the shock is infinitely strong, which is equivalent to assuming that the pressure, and therefore the internal heat energy of the gas ahead of the shock front, is negligible. In the present paper it is proposed to investigate the similarity flows which exists behind infinitely strong shock waves when the restriction of constant energy is removed, and the total energy is allowed to vary with time in the manner

$$E = E_0 t^s, \tag{1.3}$$

where E is the total energy and E_0 and s are constants. This class of flows includes the blast wave, which is given by the value $s = 0$. Furthermore, attention will be confined to positive values of s only, that is, to those cases in which the total energy increases with time. Since the motion of the gas is adiabatic, and the shock is infinitely strong, this increase can only be achieved by the pressure exerted on the gas by an expanding surface—plane, cylinder, or sphere. Thus the flow is headed by a shock front and has an expanding surface as an inner boundary. The position of this inner surface will be determined from integration of the equations of motion; accordingly a step-by-step method of integration is used, starting at the shock front and continuing into the gas until the surface is reached. Particular attention is paid to the energy of the flow: in the blast wave nearly 80 per cent. of the energy present is in the form of internal heat energy and it will be seen that one of the effects of an increase in the value of the parameter s is to reduce this ratio, until in the case of a uniformly expanding surface there is an equipartition of energy. This result is of some interest in view of recent applications of similarity solutions to problems in hypersonic flow (**5**). Throughout the subsequent working it will be assumed that the density of the undisturbed gas ahead of the advancing shock is uniform. Numerical solutions for spherically symmetric flows have been obtained by Sir Geoffrey Taylor in the two cases $s = 0$ (**2**) and $s = 3$ (**1**). The solution for cylindrically symmetric flow with $s = 0$ has been found numerically by Lin (**6**), while analytic solutions with $s = 0$ in the three cases of plane, cylindrical, and spherical flow have been noted by Sakurai (**7**).

2. Equations governing the motion

The equations describing the one-dimensional unsteady flow of a perfect gas are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{2.1}$$

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial r}+\rho\frac{\partial u}{\partial r}+\frac{ku}{r} = 0, \tag{2.2}$$

$$\frac{\partial}{\partial t}(p\rho^{-\gamma})+u\frac{\partial}{\partial r}(p\rho^{-\gamma}) = 0, \tag{2.3}$$

where u, p, ρ are the velocity, pressure, and density of the gas, and k takes the values 0, 1, 2 for the respective cases of plane, cylindrical, and spherical symmetry. In the subsequent analysis it will be convenient to replace equation (2.3) by the equivalent equation

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2+\frac{p}{\gamma-1}\right)+\frac{1}{r^k}\frac{\partial}{\partial r}\left\{r^k u\left(\tfrac{1}{2}\rho u^2+\frac{\gamma p}{\gamma-1}\right)\right\} = 0. \tag{2.4}$$

Equations (2.1), (2.2), and (2.4) are the fundamental set to be integrated, subject to boundary conditions which will be discussed below. Turning now to the choice of a similarity variable, it will be recalled that in the case of the blast wave problem this variable was

$$x = r/R, \tag{2.5}$$

where R is a function of time only and gives the position of the shock front at any instant, and this definition will also be employed in the present work. Accordingly, the velocity of expansion of the shock front is

$$V = dR/dt \tag{2.6}$$

and the shock front is represented by $x = 1$. Solutions of equations (2.1), (2.2), and (2.4) are now sought in which the velocity, density, and pressure have the forms

$$u = Vf(x), \tag{2.7}$$

$$p = \frac{\rho_0 V^2}{\gamma}g(x), \tag{2.8}$$

$$\rho = \rho_0 h(x), \tag{2.9}$$

where ρ_0 is the density of the undisturbed gas ahead of the shock front. The total energy of the flow may be written as

$$E = \int \tfrac{1}{2}\rho u^2\,d\tau+\int\frac{p}{\gamma-1}\,d\tau, \tag{2.10}$$

where $d\tau$ is an element of volume; the first integral represents the total kinetic energy of the gas, and the second integral the internal heat energy.

where α is a constant. When the density of the undisturbed gas is uniform, and it is assumed that the total energy carried by the flow remains constant, the resulting solution, first obtained by Sir Geoffrey Taylor (**2**), describes the initial stages of an intense explosion. Now the total energy in the flow is the sum of the kinetic and internal heat energies of the gas; in practice there will be losses due to dissipative effects while there will also be a gain in the internal heat energy as the shock front advances and encloses more of the quiescent gas. However, this increase has effectively been omitted because of the assumption that the shock is infinitely strong, which is equivalent to assuming that the pressure, and therefore the internal heat energy of the gas ahead of the shock front, is negligible. In the present paper it is proposed to investigate the similarity flows which exists behind infinitely strong shock waves when the restriction of constant energy is removed, and the total energy is allowed to vary with time in the manner

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where ρ_0 is the density of the undisturbed gas ahead of the shock front. The total energy of the flow may be written as

$$E=\int \tfrac{1}{2}\rho u^2\,d\tau+\int \frac{p}{\gamma-1}\,d\tau, \tag{2.10}$$

where $d\tau$ is an element of volume; the first integral represents the total kinetic energy of the gas, and the second integral the internal heat energy.

By using the assumed forms for the dependent variables this can be put in the form

$$E = \rho_0 \epsilon_k V^2 R^{k+1} \int\limits_{x_0}^{1} \left\{\tfrac{1}{2}hf^2 + \frac{g}{\gamma(\gamma-1)}\right\} x^k\,dx, \tag{2.11}$$

where $\epsilon_k = 2^k \pi^{\frac{1}{2}k(3-k)}$, and x_0 is the coordinate of the expanding surface. Since f, g, and h are functions of x only, this integral will be a function of k and γ, and (implicitly) s. Now the type of flow under consideration is one in which the energy increases with time, and the law of variation has already been stated, namely equation (1.3). From this relation and equation (2.11) it follows that the motion of the shock front is described by the equation

$$R^{\frac{1}{2}(k+1)} \frac{dR}{dt} = \left[\frac{E_0}{\epsilon_k \rho_0 J}\right]^{\frac{1}{2}} t^{\frac{1}{2}s}, \tag{2.12}$$

where

$$J = \int\limits_{x_0}^{1} \left\{\tfrac{1}{2}hf^2 + \frac{g}{\gamma(\gamma-1)}\right\} x^k\,dx. \tag{2.13}$$

Equation (2.12), on integration, yields

$$R = \left(\frac{k+3}{s+2}\right)^{2/(k+3)} \left(\frac{E_0}{\epsilon_k \rho_0 J}\right)^{1/(k+3)} t^{(s+2)/(k+3)}, \tag{2.14}$$

where it is assumed that $R = 0$ initially. From the relation (2.14) it can be seen that the value $s = k+1$ corresponds to the uniform expansion of a plane, cylinder, or sphere; the solutions of physical significance appear to be those for which s lies in the range 0 to $k+1$. The basic set of equations (2.1), (2.2), and (2.4) can now be put in the form

$$(x-f)f' = \frac{g'}{\gamma h} + \frac{s-k-1}{s+2} f, \tag{2.15}$$

$$(x-f)h' = h\left(f' + \frac{kf}{x}\right), \tag{2.16}$$

$$\left\{x^k f\left(\tfrac{1}{2}hf^2 + \frac{g}{\gamma-1}\right)\right\}' = x^{k+1}E_1'(x) + \frac{2(k+1-s)}{s+2} x^k E_1(x) \tag{2.17}$$

where

$$E_1 = \tfrac{1}{2}hf^2 + \frac{g}{\gamma(\gamma-1)},$$

and a prime denotes differentiation with respect to x. The boundary conditions must now be examined and these are discussed by Kynch (**3**); in the present notation, for very strong shocks, these are

$$f(1) = \frac{2}{\gamma+1}, \tag{2.18}$$

$$g(1) = \frac{2\gamma}{\gamma+1}, \tag{2.19}$$

$$h(1) = \frac{\gamma+1}{\gamma-1}. \tag{2.20}$$

Equations (2.15), (2.16), and (2.17) are now integrated numerically using the Runge–Kutta–Gill method starting at the shock front and continuing until a value x_0 is reached such that $f(x_0) = x_0$. This is simply the kinematic condition at the expanding surface which states that the velocity of the gas is equal to the velocity of the surface itself. In view of the similarity assumption concerning the flow, the total mass of gas between the expanding surface and the shock front should be the same as that originally contained in the volume given by $0 \leqslant x \leqslant 1$ and this fact was used as a check on the accuracy of the numerical integrations. The results of the integrations are shown in the following tables; the values of k and γ used are indicated for each table. The second column gives the value x_0 at which the surface occurs, and the third column the value of the pressure at this point. The fourth column gives the value of the integral J defined in equation (2.13); this is used in evaluating the total energy carried by the flow. The integrations were performed on the University of Illinois digital computer.

TABLE 1

$\gamma = 1{\cdot}4$. $k = 2$. *Spherical flow*

S	x_0 $(= f(x_0))$	$g(x_0)$	J
0	0	0·4264	0·4264
0·07	0·578	0·4854	0·3960
0·2	0·736	0·5748	0·3610
0·5	0·846	0·7302	0·3173
1·00	0·900	0·9079	0·2832
2·00	0·932	1·1215	0·2556
3·00	0·942	1·2450	0·2526

TABLE 2

$\gamma = 1{\cdot}4$. $k = 1$. *Cylindrical flow*

S	x_0	$g(x)_0$	J
0	0	0·4351	0·6414
0·05	0·408	0·4814	0·5900
0·07	0·468	0·4986	0·5773
0·1	0·534	0·5232	0·5607
0·2	0·664	0·5975	0·5178
0·4	0·776	0·7206	0·4660
0·6	0·828	0·8211	0·4344
1·0	0·875	0·9776	0·3992
2·0	0·915	1·2246	0·3702

TABLE 3

$\gamma = 1\cdot 4.\quad k = 0.\quad$ *Plane flow*

S	x_0	$g(x_0)$	J
0	0	0·4550	1·2174
0·05	0·221	0·5086	1·1001
0·1	0·365	0·5591	1·0206
0·2	0·536	0·6523	0·9177
0·4	0·694	0·8139	0·8094
0·5	0·736	0·8846	0·7774
0·6	0·768	0·9500	0·7500
0·8	0·808	1·0662	0·7164
1·0	0·833	1·1667	0·6958

TABLE 4

$\gamma = 1\cdot 2.\quad k = 2.\quad$ *Spherical flow*

S	x_0	$g(x_0)$	J
0	0	0·4621	0·8592
0·07	0·657	0·5055	0·7068
0·2	0·811	0·5749	0·5637
0·5	0·903	0·6999	0·4309
1·0	0·942	0·8458	0·3541
2·0	0·962	1·0251	0·2949
3·0	0·970	1·1275	0·2882

TABLE 5

$\gamma = 1\cdot 2.\quad k = 1.\quad$ *Cylindrical flow*

S	x_0	$g(x_0)$	J
0	0	0·4648	1·2892
0·05	0·501	0·5006	1·0785
0·07	0·567	0·5142	1·0203
0·1	0·637	0·5338	0·9491
0·2	0·762	0·5938	0·7932
0·4	0·857	0·6951	0·6407
0·6	0·896	0·7786	0·5644
1·0	0·929	0·9101	0·4860
2·0	0·954	1·1182	0·4251

TABLE 6

$\gamma = 1\cdot 2.\quad k = 0.\quad$ *Plane flow*

S	x_0	$g(x_0)$	J
0	0	0·4715	2·5200
0·05	0·334	0·5184	1·9722
0·1	0·508	0·5626	1·6699
0·2	0·682	0·6440	1·3439
0·4	0·814	0·7849	1·0653
0·5	0·844	0·8464	0·9959
0·6	0·866	0·9033	0·9365
0·8	0·893	1·0043	0·8640
1·0	0·909	1·0909	0·8271

TABLE 7

Values of g_0 for different values of γ

γ	*Plane*	*Cylindrical*	*Spherical*
1·0	0·50000	0·50000	0·50000
1·1	0·48361	0·48158	0·48083
1·2	0·47154	0·46476	0·46211
1·3	0·46231	0·44932	0·44401
1·4	0·45502	0·43507	0·42637
1·67	0·44149	0·40188	0·38274
2·0	0·43124	0·36788	0·33442
3·0	0·41693	0·29630	0·20233

3. Discussion of results

Although the integration of equations (2.15), (2.16), and (2.17) was performed numerically, it is of interest to note that a partial analytic solution of the problem exists. Writing equations (2.3) and (2.16) as

$$\frac{g'}{g} = \frac{1}{x-f}\left\{\gamma\left(f'+\frac{kf}{x}\right)+\frac{2(s-k-1)}{s+2}\right\}, \tag{3.1}$$

$$\frac{h'}{h} = \frac{1}{x-f}\left\{f'+\frac{kf}{x}\right\}, \tag{3.2}$$

a first integral can be obtained by multiplying (3.2) by

$$\left\{\gamma+\frac{2(s-k-1)}{(s+2)(k+1)}\right\}$$

and subtracting the result from (3.1); the resulting equation can then be integrated, yielding

$$g = C\{x^k(x-f)\}^{-\delta}h^{\gamma-\delta}, \tag{3.3}$$

where C is a constant which can be determined from the shock front boundary conditions, and

$$\delta = \frac{2(1+k-s)}{(s+2)(k+1)}.$$

Inspection of equation (2.17) shows that this equation integrates immediately if

$$\frac{2(k+1-s)}{s+2} = k+1,$$

or

$$(k+3)s = 0. \tag{3.4}$$

This leads to the analytic solution for the blast wave in a uniform atmosphere. A complete analytic solution also exists in the case when $s = 1$ and $k = 0$, which represents the flow in front of a plane piston moving with a uniform velocity into a gas which is at rest. The expressions for

the functions f, g, and h are

$$f(x) = \frac{2}{\gamma+1}, \tag{3.5}$$

$$g(x) = \frac{2\gamma}{\gamma+1}, \tag{3.6}$$

$$h(x) = \frac{\gamma+1}{\gamma-1}, \tag{3.7}$$

The value of x_0, which gives the position of the piston relative to the shock front, is

$$x_0 = 2/(\gamma+1) \tag{3.8}$$

and the expressions for the velocity, density, and pressure of the gas are

$$u = \frac{2V}{\gamma+1}, \tag{3.9}$$

$$\rho = \rho_0\frac{\gamma+1}{\gamma-1}, \tag{3.10}$$

$$p = \frac{2\rho_0 V^2}{\gamma+1}, \tag{3.11}$$

where V is the (constant) velocity of the shock front; the piston is advancing with a speed of $2V/(\gamma+1)$, so that the volume occupied by the flow steadily increases with time. The total energy of the flow contained in a volume having unit cross-sectional area is

$$\begin{aligned}\mathscr{E} &= \rho_0 V^2 R \int\limits_{2/(\gamma+1)}^{1} \left\{\tfrac{1}{2}hf^2 + \frac{g}{\gamma(\gamma-1)}\right\} dx \\ &= \rho_0 V^2 R\left(\frac{4}{\gamma^2-1}\right) \int\limits_{2/(\gamma+1)}^{1} dx = \frac{4\rho_0 V^2 R}{(\gamma+1)^2}.\end{aligned} \tag{3.12}$$

It will be noticed that the values of these two integrals are the same so that the kinetic energy is equal to the internal heat energy of the gas in this case. Furthermore, the rate at which the total energy of this volume is increasing is

$$\frac{d\mathscr{E}}{dt} = \frac{4\rho_0 V^3}{(\gamma+1)^2}. \tag{3.13}$$

Now this increase is equal to the rate $\mathscr{R}$ at which work is done on the gas by a unit area of the expanding piston; this latter quantity is simply the product of the surface pressure and the velocity of expansion of the plane, so that

$$\mathscr{R} = \frac{2\rho_0 V^2}{\gamma+1}\frac{2V}{\gamma+1} = \frac{4\rho_0 V^3}{(\gamma+1)^2}. \tag{3.14}$$

Returning to the general case in which a complete analytic solution does

not appear to exist, the rate at which energy is being fed into the flow is again equal to the product of the surface pressure and the area and the velocity of the expanding surface so that

$$\mathscr{R} = \epsilon_k (x_0 R)^k \frac{\rho_0 V^2}{\gamma} g(x_0) V x_0. \tag{3.15}$$

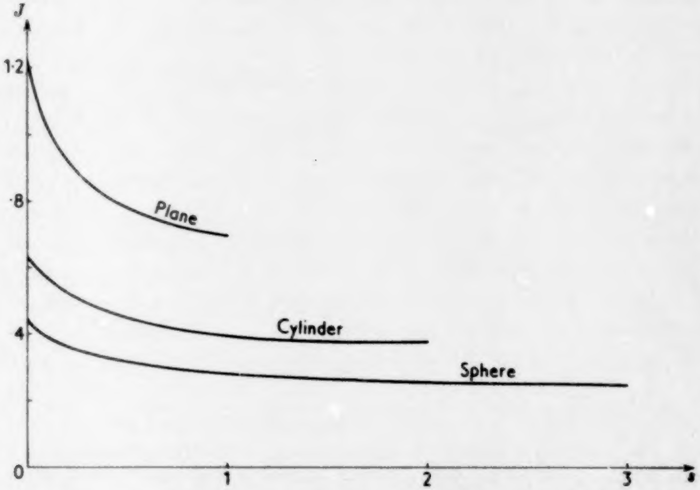

FIG. 1. The variation of the integral J, defined by equation (2.13), with the index s. In each of the three cases, only the values of s in the range $(0, k+1)$ were employed. The value of γ used in these calculations was 1·4.

Now the total energy of the flow is given by equation (1.3) and hence

$$\mathscr{R} = sE_0 t^{s-1}. \tag{3.16}$$

If these equations are now combined with (2.14) it follows that the integral J can be expressed as

$$J = \frac{(s+2)}{\gamma s(k+3)} x_0^{k+1} g(x_0) \tag{3.17}$$

provided that s does not vanish; it is found that there is good agreement between the values of J determined from the numerical integration and those obtained from (3.17), using the computed value of the surface pressure of the gas.

The manner in which J varies with s is shown in Fig. 1, where the three cases—plane, cylindrical, and spherical—have been computed using a value of 1·4 for γ. The figure shows that in each case the value of this integral decreases as s increases from zero to $k+1$. At first sight this appears to be somewhat surprising, and it is of interest to investigate

this integral a little more closely. Now the energy is comprised of two parts, the kinetic energy and the internal heat energy of the gas. When $s = 0$ the contribution of the heat energy is far greater than that of the kinetic energy; in the spherical case, Sir Geoffrey Taylor (**2**) has calculated these quantities for several values of γ. When $\gamma = 1{\cdot}4$ the kinetic energy

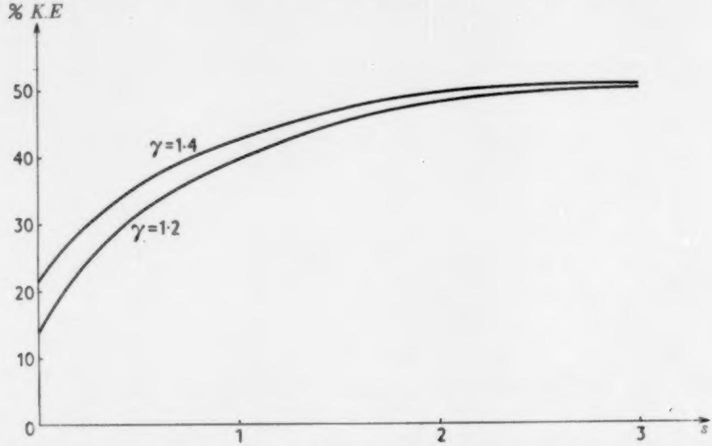

FIG. 2. The variation with s of the kinetic energy of the flow, expressed as a percentage of the total energy. The spherical case only is shown for the values of γ indicated. Note that as s approaches the limiting value of 3, the fraction approaches 50 per cent. so that there is equipartition of the energy. A similar result holds for the plane and cylindrical cases, the limiting values of s being 1 and 2 respectively.

represents only 21·8 per cent. of the total energy present and for smaller values of γ this fraction is even less. This is due to the fact that the pressure tends to a constant value as $x \to 0$, whereas both the density and velocity tend to zero at the centre so that there is very little contribution to the kinetic energy in the region $0 \leqslant x \leqslant 0{\cdot}6$. In the present case, the region in which the flow is occurring is smaller than that in the blast wave and since the pressure does not rise to an extremely high value compared with its value immediately behind the shock front, the contribution of the internal heat energy will be considerably less than in the constant energy solution. On the other hand the contribution from the kinetic energy integral rises with increasing values of s; but this increase is not sufficient to make up for the loss in the heat energy integral. Fig. 2 shows the variation of the kinetic energy expressed as a percentage of the total energy, again for spherically symmetric flows only. It will be seen that there is a sharp rise in this fraction as s increases from zero through small positive values, and then the ratio increases more slowly up to

$s = 3$ when there is approximately an equipartition of energy. The computed value is slightly over 50 per cent., but this is probably due to accumulated errors of integration. In the case of plane flow with $s = k+1$ the analytic solution displayed above shows that the two energy integrals are equal and a similar result may be expected in the spherical and cylindrical cases. It is worth noting at this point that in the well known solution of Primakoff (**8**), which describes the motion of a spherically symmetric blast wave in water, there is also an equipartition of energy. Furthermore, in the case of blast waves advancing into non-uniform atmospheres (**4**) the equipartition of energy is associated with the onset of cavitation in the flow: when no cavitation is present the kinetic energy is less than the internal heat energy in the flow, but when cavitation occurs the reverse is found to be true. For the critical flow, which separates these two regimes, there is an equipartition of energy.

In the constant energy solution, the blast wave, the pressure of the gas at the centre can be computed from the analytic solution obtained from equations (3.3) and (2.17) where s is put equal to zero. After some reduction it is found that the expression for the central pressure can be written as

$$g_0 = \gamma^{-\{(k-1)(\gamma+1)\}/\{\gamma(k+1)-(k-1)\}} 2^{-(k-1)/(k+3)} (\gamma+1)^{(k-1)(\gamma+1)/\{\gamma(k+1)-(k-1)\}} \times \\ \times \left\{\frac{-(k-1)\gamma+3k+1}{2\gamma+k-1}\right\}^{b/(\gamma-2)},$$

where $$b = \frac{(k^2+2k+5)\gamma^2+(-3k^2+2k+1)\gamma+4(k^2-1)}{(k+3)\{\gamma(k+1)-(k-1)\}}.$$

The values of g_0 can then readily be calculated and the results are shown in Table 7; some of these values are also given in Tables 1–6. The results show some points of interest: in the first place, it will be seen that in the case of isothermal flow the three values of g_0 are equal. As the value of γ increases, the central pressure gradually falls, and as would be expected the decrease is greatest for the spherically symmetric flow and least for the plane flow.

The density of the gas at the expanding surface is next examined and the partial analytic solution (3.3) is of use in this connexion. In the blast wave the density vanishes at the centre and the density gradient is also zero there. For the present case, equation (3.3) shows that, in view of the fact that the pressure remains finite at the surface,

$$h(x) \sim (x-f)^{\delta/(\gamma-\delta)}. \tag{3.18}$$

When $s = 0$ this index has the value $1/(\gamma-1)$ and when $s = k+1$ it vanishes; for intermediate values of s it is positive. Consequently, except

for the value $s = k+1$, the surface density of the gas is zero. Moreover, since the pressure is finite, the surface temperature will be infinite while the density gradient will be zero, finite, or infinite depending on whether

$$\frac{\delta}{\gamma-\delta} \lesseqgtr 1$$

or, expressing this in terms of s,

$$s \lesseqgtr \frac{2(k+1)(2-\gamma)}{4+\gamma(k+1)}. \tag{3.19}$$

When s exceeds this critical value, there is accordingly a very narrow layer on the surface across which the density falls abruptly to zero, the temperature rises to infinity while the pressure remains finite. In practice, of course, the effects of viscosity and heat conduction are of importance in this boundary layer and the similarity solution would break down at some point in this region.

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TRANSVERSE COMPONENT OF VELOCITY IN A PLANE SYMMETRICAL JET OF A COMPRESSIBLE FLUID

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[Received 17 October 1957]

SUMMARY

In the present paper, the transverse component of the velocity in the laminar motion of a plane symmetrical jet of a compressible fluid has been found, and its variation with respect to the distance from the jet axis has been studied.

1. Introduction

D. G. Toose, in a recent paper (**1**), has obtained a complete closed analytical solution for a plane compressible symmetrical jet under the usual boundary layer conditions. From his expression for the transverse component of the velocity, he has deduced that this transverse velocity is always directed towards the jet axis. He has also obtained an equation to determine the stream function and from this we can also obtain the transverse component of the velocity. We find, however, that this expression is different from his and, on a closer examination, that his first derivation of the expression for the transverse velocity is incorrect. We have found the correct expression for this and it appears that, for any given section of the jet, the transverse velocity is directed away from the axis of the jet at points near the axis. As the distance from the axis increases, this velocity component first increases and then decreases, till at some definite distance it becomes zero. After that the transverse component is directed towards the jet axis. The present conclusions appear also to be physically more plausible than the conclusions of (**1**). The streamlines and the velocity profiles turn out to be of the same type as for incompressible jets studied by Bickley (**2**). In fact, while in this degenerate case, the formula for the axial velocity component u in (**1**) reduces to the corresponding formula for u in (**2**), the formula for the transverse velocity component v in (**1**) does not reduce to the corresponding formula for v in (**2**). We verify that our formula for v reduces to Bickley's formula in the degenerate case.

We use the notation of (**1**) throughout.

2. Derivation of transverse component of velocity from the stream function

From (19), (25), and (38) of (**1**) we obtain

$$Y = y\sqrt{\left(\frac{u_j}{\nu_j LC}\right)} = \tfrac{1}{2}\sqrt{3}T_\infty X^{\frac{2}{3}}\log\left(\frac{\sqrt{6}\,X^{\frac{1}{3}}+\Psi}{\sqrt{6}\,X^{\frac{1}{3}}-\Psi}\right)+$$
$$+\tfrac{1}{2}(\gamma-1)M_j^2\left[\Psi-\Psi X^{-\frac{2}{3}}+\frac{\Psi^3X^{-1}}{18}\right]+\Psi[1-T_\infty]. \quad (1)$$

Also

$$\frac{v}{u} = \frac{\rho v}{\rho u} = -\frac{\rho_j(\partial\psi/\partial x)}{\rho_j(\partial\psi/\partial y)} = -\frac{\partial\Psi/\partial x}{\partial\Psi/\partial y} = -\frac{(1/L)\partial\Psi/\partial X}{\sqrt{(u_j/\nu_j LC)}\partial\Psi/\partial Y}. \quad (2)$$

From (1) and (2),

$$\sqrt{\left(\frac{u_j L}{\nu_j C}\right)}\frac{v}{u} = \frac{T_\infty}{X^{\frac{1}{3}}}\left[\frac{1}{\sqrt{3}}\log\frac{\sqrt{6}+\zeta}{\sqrt{6}-\zeta}-\frac{\sqrt{2}\,\zeta}{6-\zeta^2}\right]+\tfrac{1}{2}(\gamma-1)M_j^2\frac{1}{3X}\zeta\left[1-\frac{\zeta^2}{6}\right]. \quad (3)$$

But, from (**1**),

$$u = u_j X^{-\frac{1}{3}}(1-\tfrac{1}{6}\zeta^2); \quad (4)$$

hence
$$v = \sqrt{\left(\frac{u_j\nu_j C}{L}\right)}\left\{\frac{T_\infty}{X^{\frac{2}{3}}}\left[\frac{1}{\sqrt{3}}\left(1-\frac{\zeta^2}{6}\right)\log\frac{\sqrt{6}+\zeta}{\sqrt{6}-\zeta}-\frac{\sqrt{2}}{6}\zeta\right]+\right.$$
$$\left.+\tfrac{1}{2}(\gamma-1)M_j^2\frac{1}{3X^{\frac{4}{3}}}\zeta\left(1-\frac{\zeta^2}{6}\right)^2\right\}. \quad (5)$$

On the other hand, from (37) and (40) of (**1**), we get

$$v = -\sqrt{\left(\frac{u_j\nu_j C}{L}\right)}\tfrac{1}{3}\zeta X^{-\frac{2}{3}}\left\{T_\infty-\tfrac{1}{2}(\gamma-1)M_j^2X^{-\frac{2}{3}}\left(1-\frac{\zeta^2}{6}\right)^2+\right.$$
$$\left.+\{1+\tfrac{1}{2}(\gamma-1)M_j^2-T_\infty\}X^{-\frac{1}{3}}\left(1-\frac{\zeta^2}{6}\right)\right\}. \quad (6)$$

Expressions (5) and (6) are obviously different, and we examine the reason for this discrepancy and its consequences in the next section.

3. Earlier derivation of v examined

Since
$$d\psi = \left(\frac{\partial\psi}{\partial x}\right)_\zeta dx+\left(\frac{\partial\psi}{\partial\zeta}\right)_x d\zeta = \left(\frac{\partial\psi}{\partial y}\right)_x dy+\left(\frac{\partial\psi}{\partial x}\right)_y dx$$

it follows that

$$\left[\tfrac{1}{3}\zeta\sqrt{\left(\frac{u_j\nu_j C}{L}\right)}X^{-\frac{2}{3}}+X^{\frac{1}{3}}\sqrt{(u_j\nu_j LC)}\left(\frac{\partial\zeta}{\partial x}\right)_y\right]dx+X^{\frac{1}{3}}\sqrt{(u_j\nu_j LC)}\left(\frac{\partial\zeta}{\partial y}\right)_x dy$$
$$= \frac{\rho u}{\rho_j}dy-\frac{\rho v}{\rho_j}dx. \quad (7)$$

From (7),
$$v = -\frac{1}{\Omega}\left[\tfrac{1}{3}\zeta\sqrt{\left(\frac{u_j v_j C}{L}\right)}X^{-\frac{2}{3}}+X^{\frac{1}{3}}\sqrt{(u_j v_j LC)}\left(\frac{\partial\zeta}{\partial x}\right)_y\right] \tag{8}$$
and
$$u = \frac{1}{\Omega}\left[X^{\frac{1}{3}}\sqrt{(u_j v_j LC)}\left(\frac{\partial\zeta}{\partial y}\right)_x\right]. \tag{9}$$

Instead of (8), (**1**) gives
$$v = -\frac{\zeta}{3\Omega}\sqrt{\left(\frac{u_j v_j C}{L}\right)}X^{-\frac{2}{3}}, \tag{10}$$
which is obviously incorrect. In (**1**), (10) has been used to determine v and the boundary conditions for solving the basic differential equation and to deduce that the transverse velocity in the jet is directed towards the jet axis. We examine the necessary modifications below.

From (13), (16), (18), (19), and (25) of (**1**)
$$\left(\frac{\partial\zeta}{\partial x}\right)_y = \frac{1}{\sqrt{(u_j v_j LC)}}\left[-\frac{\rho v}{\rho_j}X^{-\frac{1}{3}}-\tfrac{1}{3}\Psi X^{-\frac{4}{3}}\frac{1}{L}\right]. \tag{11}$$
Substituting (11) in (8) gives
$$\zeta = \Psi X^{-\frac{1}{3}}. \tag{12}$$
Using (19) and (25) of (**1**) we see that (8) is an identity which cannot determine v. The only way of determining v is therefore through the stream function, as has been done above.

In (**1**), (10) has been used to deduce the boundary condition
$$\zeta = 0 \quad \text{when} \quad y = 0. \tag{13}$$
Fortunately (13) can also be deduced from symmetry considerations, for by symmetry $y = 0$ is a streamline and since numbering of streamlines is arbitrary, we can take
$$\Psi = 0 \quad \text{when} \quad y = 0. \tag{14}$$
(13) then follows from (12) and (14).

From (9), we also get
$$\left(\frac{\partial y}{\partial\zeta}\right)_x = \sqrt{(u_j v_j LC)}\frac{X^{\frac{1}{3}}\rho_j}{\rho u} = \frac{\sqrt{(v_j LC)}}{u_j}X^{\frac{2}{3}}\frac{T}{F(\zeta)}. \tag{15}$$
Integrating,
$$y = \sqrt{\left(\frac{v_j LC}{u_j}\right)}X^{\frac{2}{3}}\int\limits_a^\zeta \frac{T}{F(\zeta)}\,d\zeta+\phi(X), \tag{16}$$
where a is an arbitrary constant and $\phi(X)$ an arbitrary function of X. However, since from (13) we have $y = 0$ when $\zeta = 0$, it follows that
$$\phi(X) = 0, \qquad a = 0$$
and so
$$y = \sqrt{\left(\frac{v_j LC}{u_j}\right)}X^{\frac{2}{3}}\int\limits_0^\zeta \frac{T}{F(\zeta)}\,d\zeta. \tag{17}$$
This is the same as the corresponding equation in (**1**), but it is seen to be

true, not only when X is held constant, but also when its variation is taken into account.

4. Discussion of variation of transverse component

In (5), the coefficient of $1/X^{\frac{1}{3}}$ is always positive, while the coefficient of $1/X^{\frac{2}{3}}$ is positive for small values of ζ and takes negative values for larger values of ζ. Also $v = 0$ when $\zeta = 0$ and v approaches a finite negative value when $\zeta \to \sqrt{6}$. Again, from (15), $(\partial y/\partial\zeta)_x$ is always positive, so that for every fixed x, as y increases ζ increases. Thus, for every x, there is a value of y such that up to that value, the transverse velocity is directed away from the jet axis and after that the transverse velocity is directed towards the jet axis.

The equation of the curve which divides the regions in which transverse velocity is directed towards or away from the jet axis is given by

$$y\sqrt{\left(\frac{u_j}{\nu_j LC}\right)} = T_\infty X^{\frac{2}{3}}\sqrt{\tfrac{3}{2}}\log\left(\frac{\sqrt{6}+\zeta}{\sqrt{6}-\zeta}\right) - \tfrac{1}{2}(\gamma-1)M_j^2\left(\zeta-\frac{\zeta^3}{18}\right) + X^{\frac{1}{3}}\zeta(T_0-T_\infty), \tag{18}$$

$$0 = \frac{T_\infty}{X^{\frac{2}{3}}}\left[\frac{1}{\sqrt{3}}\left(1-\frac{\zeta^2}{6}\right)\log\left(\frac{\sqrt{6}+\zeta}{\sqrt{6}-\zeta}\right) - \frac{\sqrt{2}}{6}\zeta\right] + \tfrac{1}{2}(\gamma-1)M_j^2\frac{1}{3X^{\frac{1}{3}}}\zeta\left(1-\frac{\zeta^2}{6}\right)^2, \tag{19}$$

where ζ is the parameter.

For a fixed X, putting $(\partial v/\partial\zeta) = 0$, we get the value of ζ for which v has an extreme value and then (1) and (11) determine the corresponding value of y.

5. A degenerate case

In the degenerate case, when $t_j = t_\infty$ and the orifice velocity u_j is very small, proceeding as in (1) we find

$$\frac{\zeta}{\sqrt{6}} = \tanh\xi, \tag{20}$$

where

$$\xi = \frac{\alpha y}{3x^{\frac{2}{3}}}, \qquad \alpha = \left(\frac{9M}{16\rho_j\nu_j^2}\right)^{\frac{1}{3}}. \tag{21}$$

In this case (5) gives

$$\frac{u}{u_j} = X^{-\frac{1}{3}}\left(1-\frac{\zeta^2}{6}\right) = X^{-\frac{1}{3}}\operatorname{sech}^2\xi, \tag{22}$$

$$v = \frac{u_j\nu_j C}{L}\frac{T_\infty}{\sqrt{3}}\frac{1}{X^{\frac{2}{3}}}\{2\xi\operatorname{sech}^2\xi - \tanh\xi\}, \tag{23}$$

which agrees with Bickley's formulae.

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ON THE THEORY OF ANISOTROPIC OBSTACLES IN WAVEGUIDES†

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[Received 26 September 1957]

SUMMARY

Variational principles for the approximate computation of the elements of the scattering matrix for anisotropic obstacles in waveguides are presented.

1. Introduction

THE effect of a ferrite obstacle in a waveguide on the propagating modes of the guide may be described in terms of the scattering matrix, which relates the reflected and incident amplitudes of the propagating modes at suitably chosen reference planes. The calculation of the elements of the scattering matrix involves a knowledge of the electromagnetic field within the obstacle (equations 16–21), which has to be obtained from a solution of Maxwell's equations everywhere within the guide subject to appropriate boundary conditions.

The exact solution of this problem is a formidable task if not an impossible one. Quite naturally therefore, we seek approximate solutions. This is the purpose of this paper.

The method which we develop here is an extension of the work of J. Schwinger (**1**) on isotropic obstacles in waveguides to anisotropic obstacles. With the introduction of an appropriate dyadic Green's function we are able to obtain a formal solution to the problem in terms of several integrals involving the field vectors within the obstacle. The resulting integral equations are by no means easier to solve. It is possible, however, to set up variational expressions for the elements of the scattering matrix, which are stationary with respect to first-order variations of the fields about their true value.

Consequently, we have a very powerful method for obtaining approximate solutions for the elements of the scattering matrix.

† The research reported in this document was supported jointly by the Army, Navy, and Air Force under contract with Massachusetts Institute of Technology.

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2. Integral equations for the electromagnetic field and elements of the scattering matrix

We consider the problem of an obstacle with tensor electromagnetic properties within a waveguide. The electromagnetic fields within the guide, assuming a time factor $e^{j\omega t}$, satisfy the differential equations

$$\left.\begin{aligned} \nabla\times\mathbf{E} &= -j\omega\mu_0\mathbf{H}-j\omega\boldsymbol{\mu}'.\mathbf{H} \\ \text{and}\qquad \nabla\times\mathbf{H} &= j\omega\epsilon_0\mathbf{E}+j\omega\boldsymbol{\epsilon}'.\mathbf{E} \end{aligned}\right\}; \tag{1}$$

ϵ_0 and μ_0 are the electric permittivity and magnetic permeability of the empty guide. The tensor magnetic permeability $\boldsymbol{\mu}$ and the tensor electric permittivity $\boldsymbol{\epsilon}$ of the obstacle are given by

$$\boldsymbol{\mu} = \boldsymbol{\mu}'+\mu_0\mathbf{I}, \qquad \boldsymbol{\epsilon} = \boldsymbol{\epsilon}'+\epsilon_0\mathbf{I},$$

where $\mathbf{I}$ is the unit dyadic; $\boldsymbol{\epsilon}'$ and $\boldsymbol{\mu}'$ in equation (1) are to be taken zero outside the obstacle.

The wave equations satisfied by the electromagnetic field vectors are

$$\nabla\times\nabla\times\mathbf{E}-k^2\mathbf{E} = \omega^2\mu_0\,\boldsymbol{\epsilon}'.\mathbf{E}-j\omega\nabla\times(\boldsymbol{\mu}'.\mathbf{H}) \tag{2a}$$

and

$$\nabla\times\nabla\times\mathbf{H}-k^2\mathbf{H} = \omega^2\epsilon_0\,\boldsymbol{\mu}'.\mathbf{H}+j\omega\nabla\times(\boldsymbol{\epsilon}'.\mathbf{E}), \tag{2b}$$

where $k^2 = \omega^2\epsilon_0\mu_0$.

To solve equation (2a) we utilize the electric dyadic Green's function $\mathbf{N}(\boldsymbol{\rho},z|\boldsymbol{\rho}',z')$, which satisfies the differential equation

$$\nabla\times\nabla\times\mathbf{N}-k^2\mathbf{N} = \mathbf{I}\delta(\mathbf{r}-\mathbf{r}') \tag{3}$$

and the boundary condition

$$\mathbf{n}\times\mathbf{N}(\boldsymbol{\rho},z\,|\,\boldsymbol{\rho}',z') = 0$$

when $\boldsymbol{\rho} = x\mathbf{i}+y\mathbf{j}$ lies on the boundary of the guide. At the reference planes $\mathbf{N}$ must represent radiated waves moving in the increasing $|z-z'|$ direction. If the electric field within the obstacle is non-divergent then we shall, in addition, require the electric dyadic Green's function to be non-divergent too. Such is the case, for example, for ferrite obstacles for which $\boldsymbol{\epsilon} = \epsilon\mathbf{I}$.

The non-divergent part of the dyadic Green's function $\mathbf{N}_1$ is expansible in terms of the electric fields of the usual TE and TM modes of the guide. If $\mathbf{E}_n^{(1)}e^{-j\gamma_n z}$ is the electric field of the nth mode moving in the increasing z-direction, and $\mathbf{E}_n^{(2)}e^{j\gamma_n z}$ the electric field of the wave moving in the decreasing z-direction, we find that

$$\mathbf{N}_1(\boldsymbol{\rho},z\,|\,\boldsymbol{\rho}',z') = \sum_{n=1}^{\infty} A_n\times\begin{cases}\mathbf{E}_n^{(1)}(\boldsymbol{\rho})\mathbf{E}_n^{(2)}(\boldsymbol{\rho}')e^{-j\gamma_n(z-z')} & (z>z')\\ \mathbf{E}_n^{(2)}(\boldsymbol{\rho})\mathbf{E}_n^{(1)}(\boldsymbol{\rho}')e^{j\gamma_n(z-z')} & (z<z'),\end{cases} \tag{4}$$

where

$$A_n = \frac{-j}{2\gamma_n}\times\begin{cases} k_{cn}^2/\omega^2\mu_0^2 \int H_z^2\,dS & \text{(for TE modes)} \\ k_{cn}^2/k^2 \int E_z^2\,dS & \text{(for TM modes)}\end{cases} \quad (k_{cn}^2 = k^2-\gamma_n^2). \tag{5}$$

The curl-less part of the dyadic Green's function $\mathbf{N}_2$, which we should expect to be derivable from the gradient of a vector function, satisfies the differential equation

$$-k^2\nabla.\mathbf{N}_2 = \nabla\delta(\mathbf{r}-\mathbf{r}').$$

It is readily shown (**2**) that

$$\mathbf{N}_2(\mathbf{r}|\mathbf{r}') = -\frac{1}{k^2}\nabla\nabla' G(\mathbf{r}|\mathbf{r}'), \tag{6}$$

where $G(\mathbf{r}|\mathbf{r}')$ is the scalar Green's function satisfying the differential equation

$$\nabla^2 G = -\delta(\mathbf{r}-\mathbf{r}'). \tag{7}$$

In order that $\mathbf{n}\times\mathbf{N}_2$ should satisfy the boundary conditions, $G(\mathbf{r}|\mathbf{r}')$ should be zero on the boundary of the guide and likewise be exponentially damped with increasing $|z-z'|$. Such a scalar Green's function is given by

$$G(\mathbf{r}|\mathbf{r}') = \sum_{n=1}^{\infty}\frac{1}{2k_n}\phi_n(x,y)\phi_n(x',y')e^{-k_n|z-z'|}, \tag{8}$$

where the ϕ_n's are the normalized eigensolutions of the differential equation

$$\frac{\partial^2\phi_n}{\partial x^2}+\frac{\partial^2\phi_n}{\partial y^2}+k_n^2\phi_n^2 = 0$$

with zero boundary conditions. The k_n's are the eigenvalues.

To obtain the integral equations for $\mathbf{E}$, we take the scalar product of equation (2a) with $\mathbf{N}_a = \mathbf{N}.\mathbf{a}$, where $\mathbf{a}$ is an arbitrary vector introduced in order to simplify the handling of the dyadic Green's function, and subtract from the result the scalar product of equation (3) with $\mathbf{E}$ from the left and $\mathbf{a}$ from the right. Integration of the result over the volume between the reference planes, chosen far enough away from the obstacle so that all the cut-off modes excited by the obstacle are of negligible amplitude, yields

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{\text{inc}}(\mathbf{r})+\omega^2\mu_0\int \mathbf{N}^T(\mathbf{r}'|\mathbf{r}).\boldsymbol{\epsilon}'.\mathbf{E}(\mathbf{r}')\,d\tau'-$$
$$-j\omega\int[\nabla'\times\mathbf{N}(\mathbf{r}'|\mathbf{r})]^T.[\boldsymbol{\mu}'.\mathbf{H}(\mathbf{r}')]\,d\tau', \tag{9}$$

where $\mathbf{E}_{\text{inc}}$ is the electric field of the wave incident on the obstacle, and $\mathbf{N}^T$ and $(\nabla\times\mathbf{N})^T$ are the transpose of $\mathbf{N}$ and $(\nabla\times\mathbf{N})$ respectively. To obtain the magnetic field, we may utilize the above equation and Maxwell's equations, or repeat the above-mentioned steps utilizing equation

(2b) and the magnetic dyadic Green's function $\mathbf{M} = \mathbf{M}_1+\mathbf{M}_2$. The non-divergent part is

$$\mathbf{M}_1(\boldsymbol{\rho}, z|\boldsymbol{\rho}', z') = \sum_{n=1}^{\infty} B_n \times \begin{cases} \mathbf{H}_n^{(1)}(\boldsymbol{\rho})\mathbf{H}_n^{(2)}(\boldsymbol{\rho}')e^{-j\gamma_n(z-z')} & (z > z') \\ \mathbf{H}_n^{(2)}(\boldsymbol{\rho})\mathbf{H}_n^{(1)}(\boldsymbol{\rho}')e^{j\gamma_n(z-z')} & (z < z'), \end{cases}$$

where $$B_n = -\frac{j}{2\gamma_n} \times \begin{cases} k_{cn}^2/k^2 \int H_z^2\, dS & \text{(for TE modes)} \\ k_{cn}^2/\omega^2\epsilon_0^2 \int E_z^2\, dS & \text{(for TM modes).} \end{cases}$$

The curl-less part is $$\mathbf{M}_2 = -\frac{1}{k^2}\nabla\nabla' g(\mathbf{r}|\mathbf{r}'),$$

where $$g(\mathbf{r}|\mathbf{r}') = \sum_{m=1}^{\infty} \frac{1}{2k_m}\psi_m(x,y)\psi_m(x',y')e^{-k_m|z-z'|}.$$

The ψ's satisfy the differential equation

$$\frac{\partial^2\psi_m}{\partial x^2}+\frac{\partial^2\psi_m}{\partial y^2}+k_m^2\psi_m = 0$$

and the boundary condition $\mathbf{n}.\nabla\psi_m = 0$ at the boundary of the guide, since M satisfies the boundary condition $\mathbf{n}.(\nabla\times\mathbf{M}) = 0$. We find

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_{\text{inc}}(\mathbf{r})+\omega^2\epsilon_0 \int \mathbf{M}^T(\mathbf{r}'|\mathbf{r}).\boldsymbol{\mu}'.\mathbf{H}(\mathbf{r}')\, d\tau' + \\ +j\omega \int [\nabla'\times\mathbf{M}(\mathbf{r}'|\mathbf{r})]^T.[\boldsymbol{\epsilon}'.\mathbf{E}(\mathbf{r}')]\, d\tau'. \quad (10)$$

To obtain explicit expressions for the elements of the scattering matrix we take the limit of equation (9) as the observation point approaches the reference planes. Near the first reference plane ($z \to -\infty$), assuming for simplicity that only the first mode propagates within the guide, we obtain, utilizing the expanded form of the dyadic Green's function,

$$\mathbf{E}(\mathbf{r}) \to \mathbf{E}_{\text{inc}}+A_1 J_1 \mathbf{E}_1^{(2)}e^{j\gamma_1 z}, \quad (11)$$

where $$J_1 = \omega^2\mu_0 \int (\mathbf{E}_1^{(1)}.\boldsymbol{\epsilon}'.\mathbf{E}-\mathbf{H}_1^{(1)}.\boldsymbol{\mu}'.\mathbf{H})e^{-j\gamma_1 z'}\, d\tau'. \quad (12)$$

Near the second reference plane ($z \to +\infty$) we similarly find

$$\mathbf{E}(\mathbf{r}) \to \mathbf{E}_{\text{inc}}+A_1 J_2 \mathbf{E}_1^{(1)}e^{-j\gamma_1 z}, \quad (13)$$

where $$J_2 = \omega^2\mu_0 \int (\mathbf{E}_1^{(2)}.\boldsymbol{\epsilon}'.\mathbf{E}-\mathbf{H}_1^{(2)}.\boldsymbol{\mu}'.\mathbf{H})e^{j\gamma_1 z'}\, d\tau'. \quad (14)$$

Setting $\mathbf{E} = \mathbf{E}^{(1)}(\mathbf{r})$

when $\mathbf{E}_{\text{inc}} = \mathbf{E}_1^{(1)}e^{-j\gamma_1 z}$,

and $\mathbf{E}(\mathbf{r}) = \mathbf{E}^{(2)}(\mathbf{r})$

when $\mathbf{E}_{\text{inc}} = \mathbf{E}_1^{(2)}e^{+j\gamma_1 z}$,

we find near the first plane that

$$\left.\begin{aligned} \mathbf{E}^{(1)}(\mathbf{r}) &= \mathbf{E}_1^{(1)}e^{-j\gamma_1 z}+A_1 J_{11}\mathbf{E}_1^{(2)}e^{j\gamma_1 z} \\ \text{and} \quad \mathbf{E}^{(2)}(\mathbf{r}) &= \mathbf{E}_1^{(2)}e^{j\gamma_1 z}+A_1 J_{12}\mathbf{E}_1^{(2)}e^{+j\gamma_1 z} \end{aligned}\right\}, \tag{15}$$

where

$$J_{11} = \omega^2\mu_0 \int (\mathbf{E}_1^{(1)}.\boldsymbol{\epsilon}'.\mathbf{E}^{(1)}-\mathbf{H}_1^{(1)}.\boldsymbol{\mu}'.\mathbf{H}^{(1)})e^{-j\gamma_1 z'}\,d\tau' \tag{16}$$

and

$$J_{12} = \omega^2\mu_0 \int (\mathbf{E}_1^{(1)}.\boldsymbol{\epsilon}'.\mathbf{E}^{(2)}-\mathbf{H}_1^{(1)}.\boldsymbol{\mu}'.\mathbf{H}^{(2)})e^{-j\gamma_1 z'}\,d\tau'. \tag{17}$$

Similarly, near the second plane we obtain

$$\left.\begin{aligned} \mathbf{E}^{(1)}(\mathbf{r}) &= \mathbf{E}_1^{(1)}e^{-j\gamma_1 z}+A_1 J_{21}\mathbf{E}_1^{(1)}e^{-j\gamma_1 z} \\ \text{and} \quad \mathbf{E}^{(2)}(\mathbf{r}) &= \mathbf{E}_1^{(2)}e^{j\gamma_1 z}+A_1 J_{22}\mathbf{E}_1^{(1)}e^{-j\gamma_1 z} \end{aligned}\right\}, \tag{18}$$

$$\left.\begin{aligned} \text{where} \quad J_{21} &= \omega^2\mu_0 \int (\mathbf{E}_1^{(2)}.\boldsymbol{\epsilon}'.\mathbf{E}^{(1)}-\mathbf{H}_1^{(2)}.\boldsymbol{\mu}'.\mathbf{H}^{(1)})e^{j\gamma_1 z'}\,d\tau' \\ \text{and} \quad J_{22} &= \omega^2\mu_0 \int (\mathbf{E}_1^{(2)}.\boldsymbol{\epsilon}'.\mathbf{E}^{(2)}-\mathbf{H}_1^{(2)}.\boldsymbol{\mu}'.\mathbf{H}^{(2)})e^{j\gamma_1 z'}\,d\tau' \end{aligned}\right\} \tag{19}$$

or

$$J_{nm} = \omega^2\mu_0 \int [\mathbf{E}_1^{(n)}.\boldsymbol{\epsilon}'.\mathbf{E}^{(m)}-\mathbf{H}_1^{(n)}.\boldsymbol{\mu}'.\mathbf{H}^{(m)}]e^{(-1)^n j\gamma_1 z'}\,d\tau' \quad (n,m = 1,2). \tag{20}$$

The effect of the obstacle on the propagating mode of the guide is thus given in terms of the four J_{nm}, which are related to the elements of the scattering matrix. We note that

$$\left.\begin{aligned} S_{11} &= A_1 J_{11} = r_1 \\ S_{12} &= 1+A_1 J_{12} = 1+t_2 \\ S_{21} &= 1+A_1 J_{21} = 1+t_1 \\ \text{and} \quad S_{22} &= A_1 J_{22} = r_2 \end{aligned}\right\}, \tag{21}$$

where the r's and t's are the reflection and transmission coefficients of the obstacle. The subscripts (1, 2) on the reflection coefficients r and the transmission coefficients t, indicate whether the incident wave is coming from the left, $z = -\infty$, or the right, $z = \infty$. The exact solution of integral equations (9) and (10) satisfied by the electromagnetic field required for the evaluation of the elements of the scattering matrix, is a formidable task. Fortunately, however, it is possible to construct stationary expressions for the elements of the J matrix, thus enabling us to obtain good approximate results for these elements whenever a good trial function suggests itself.

3. Variational principle for J_{nm}

To construct a variational principle from which integral equations (9) and (10) for the electromagnetic field within the obstacle are derivable, it is necessary to define an adjoint electromagnetic field. This field we

choose as the solution of the differential equation

$$\left.\begin{aligned} \nabla \times \mathbf{E}^\dagger &= j\omega \mathbf{H}^\dagger . \boldsymbol{\mu} \\ \text{and} \qquad \nabla \times \mathbf{H}^\dagger &= -j\omega \mathbf{E}^\dagger . \boldsymbol{\epsilon} \end{aligned}\right\}, \tag{22}$$

and assume the adjoint field to have an $e^{-j\omega t}$ time dependence.

In some problems we shall find the adjoint fields, $\mathbf{E}^\dagger$ and $\mathbf{H}^\dagger$, to be simply related to the fields $\mathbf{E}$ and $\mathbf{H}$. For example, in the case of non-dissipative obstacles, where $\boldsymbol{\epsilon}$ and $\boldsymbol{\mu}$ are hermitian (3),

$$\mathbf{E}^\dagger = \mathbf{E}^* \quad \text{and} \quad \mathbf{H}^\dagger = \mathbf{H}^*.$$

For lossy ferrites, if we take Lax's (4) form of the tensor magnetic permeability, having its diagonal elements even in ω, and its off-diagonal elements odd in ω,

$$\mathbf{H}^\dagger(\mathbf{r},\omega) = \mathbf{H}(\mathbf{r},-\omega) \quad \text{and} \quad \mathbf{E}^\dagger(\mathbf{r},\omega) = \mathbf{E}(\mathbf{r},-\omega).$$

To solve for the adjoint electric field, we introduce the adjoint electric dyadic Green's function, $\mathbf{N}^\dagger(\mathbf{r}'|\mathbf{r})$, which we find is equal to

$$\mathbf{N}(\mathbf{r}|\mathbf{r}')^T = \mathbf{N}(\mathbf{r}'|\mathbf{r}).$$

Analogous to finding the integral equation for $\mathbf{E}(\mathbf{r})$, we obtain

$$\mathbf{E}^\dagger(\mathbf{r}') = \mathbf{E}^\dagger_{\mathrm{inc}}(\mathbf{r}') + \omega^2\mu_0 \int \mathbf{E}^\dagger . \boldsymbol{\epsilon}' . \mathbf{N}(\mathbf{r}'|\mathbf{r})^T \, d\tau + \\ + j\omega \int \mathbf{H}^\dagger . \boldsymbol{\mu}' . [\nabla \times \mathbf{N}(\mathbf{r}|\mathbf{r}')] \, d\tau \tag{23}$$

and

$$\mathbf{H}^\dagger(\mathbf{r}') = \mathbf{H}^\dagger_{\mathrm{inc}}(\mathbf{r}') + \omega^2\epsilon_0 \int (\mathbf{H}^\dagger . \boldsymbol{\mu}') . \mathbf{M}(\mathbf{r}'|\mathbf{r})^T \, d\tau - \\ - j\omega \int (\mathbf{E}^\dagger . \boldsymbol{\epsilon}') . [\nabla \times \mathbf{M}(\mathbf{r}|\mathbf{r}')] \, d\tau, \tag{24}$$

having used the reciprocity relation

$$\nabla \times \mathbf{N}(\mathbf{r}|\mathbf{r}') = [\nabla' \times \mathbf{M}(\mathbf{r}'|\mathbf{r})]^T.$$

Once again it is necessary to distinguish the solutions for different incoming waves. We should note, however, that for the adjoint case,

$$\mathbf{E}_1^{(1)} e^{-j\gamma_1 z} \quad \text{and} \quad -\mathbf{H}_1^{(1)} e^{-j\gamma_1 z}$$

are the fields for waves incident from the right ($z = \infty$), while

$$\mathbf{E}_1^{(2)} e^{j\gamma_1 z} \quad \text{and} \quad -\mathbf{H}_1^{(2)} e^{j\gamma_1 z}$$

are the fields for waves incident from the left ($z = -\infty$). Near the first reference plane we thus find

$$\mathbf{E}^{(1)\dagger}(\mathbf{r}') = \mathbf{E}_1^{(1)} e^{-j\gamma_1 z'} + A_1 J_{12}^\dagger E_1^{(1)} e^{-j\gamma_1 z'}$$

and

$$\mathbf{E}^{(2)\dagger}(\mathbf{r}') = \mathbf{E}_1^{(2)} e^{j\gamma_1 z'} + A_1 J_{22}^\dagger E_1^{(1)} e^{-j\gamma_1 z'}$$

while near the second reference plane

$$\mathbf{E}^{(1)\dagger}(\mathbf{r}') = \mathbf{E}_1^{(1)}e^{-j\gamma_1 z'} + A_1\, J_{11}^{\dagger}\, \mathbf{E}_1^{(2)} e^{j\gamma_1 z'}$$

and
$$\mathbf{E}^{(2)\dagger}(\mathbf{r}') = \mathbf{E}_1^{(2)}e^{j\gamma_1 z'} + A_1\, J_{21}^{\dagger}\, \mathbf{E}_1^{(2)} e^{j\gamma_1 z'},$$

where

$$J_{mn}^{\dagger} = \omega^2\mu_0 \int [\mathbf{E}^{(m)\dagger}.\boldsymbol{\epsilon}'.\mathbf{E}_1^{(n)} + \mathbf{H}^{(m)\dagger}.\boldsymbol{\mu}'.\mathbf{H}_1^{(n)}]e^{(-1)^n j\gamma_1 z}\, d\tau.$$

Utilizing integral equations (9), (10), (23), and (24) we can show that

$$J_{mn} = J_{mn}^{\dagger}.$$

To show this, multiply equation (9) by $(\mathbf{E}^{(m)\dagger}.\boldsymbol{\epsilon}')$ and equation (10) by $(\mathbf{H}^{(m)\dagger}.\boldsymbol{\mu}')$, to obtain

$$\int \mathbf{E}^{(m)\dagger}.\boldsymbol{\epsilon}'.\mathbf{E}^{(n)}\, d\tau = \int \mathbf{E}^{(m)\dagger}.\boldsymbol{\epsilon}'.\mathbf{E}_1^{(n)}e^{(-1)^n j\gamma_1 z}\, d\tau +$$
$$+\omega^2\mu_0 \iint \mathbf{E}^{(m)\dagger}.\boldsymbol{\epsilon}'.\mathbf{N}^T(\mathbf{r}'|\mathbf{r}).\boldsymbol{\epsilon}'.\mathbf{E}^{(n)}\, d\tau d\tau' -$$
$$-j\omega \iint \mathbf{E}^{(m)\dagger}.\boldsymbol{\epsilon}'.[\nabla\times\mathbf{M}(\mathbf{r}|\mathbf{r}')].\boldsymbol{\mu}'.\mathbf{H}^{(n)}\, d\tau d\tau'$$

and

$$\int \mathbf{H}^{(m)\dagger}.\boldsymbol{\mu}'.\mathbf{H}^{(n)}\, d\tau = \int \mathbf{H}^{(m)\dagger}.\boldsymbol{\mu}'.\mathbf{H}_1^{(n)}e^{(-1)^n j\gamma_1 z}\, d\tau +$$
$$+\omega^2\epsilon_0 \iint \mathbf{H}^{(m)\dagger}.\boldsymbol{\mu}'.\mathbf{M}^T(\mathbf{r}'|\mathbf{r}).\boldsymbol{\mu}'.\mathbf{H}^{(n)}\, d\tau d\tau' +$$
$$+j\omega \int \mathbf{H}^{(m)\dagger}.\boldsymbol{\mu}'.[\nabla\times\mathbf{N}(\mathbf{r}|\mathbf{r}')].\boldsymbol{\epsilon}'.\mathbf{E}^{(n)}\, d\tau d\tau'.$$

Similarly from equations (23) and (24) we find

$$\int \mathbf{E}^{(m)\dagger}.\boldsymbol{\epsilon}'.\mathbf{E}^{(n)}\, d\tau = \int \mathbf{E}_1^{(m)}.\boldsymbol{\epsilon}'.\mathbf{E}^{(n)}e^{(-1)^m j\gamma_1 z}\, d\tau +$$
$$+\omega^2\mu_0 \int \mathbf{E}^{(m)\dagger}.\boldsymbol{\epsilon}'.\mathbf{N}^T(\mathbf{r}'|\mathbf{r}).\boldsymbol{\epsilon}'.\mathbf{E}^{(n)}\, d\tau d\tau' +$$
$$+j\omega \int \mathbf{H}^{(m)\dagger}.\boldsymbol{\mu}'.[\nabla\times\mathbf{N}(\mathbf{r}|\mathbf{r}')].\boldsymbol{\epsilon}'.\mathbf{E}^{(n)}\, d\tau d\tau'$$

and

$$\int \mathbf{H}^{(m)\dagger}.\boldsymbol{\mu}'.\mathbf{H}^{(n)}\, d\tau = -\int \mathbf{H}_1^{(m)}.\boldsymbol{\mu}'.\mathbf{H}^{(n)}e^{(-1)^m j\gamma_1 z}\, d\tau +$$
$$+\omega^2\epsilon_0 \int \mathbf{H}^{(m)\dagger}.\boldsymbol{\mu}'.\mathbf{M}^T(\mathbf{r}'|\mathbf{r}).\boldsymbol{\mu}'.\mathbf{H}^{(n)}\, d\tau d\tau' -$$
$$-j\omega \int \mathbf{E}^{(m)\dagger}.\boldsymbol{\epsilon}'.[\nabla\times\mathbf{M}(\mathbf{r}|\mathbf{r}')].\boldsymbol{\mu}'.\mathbf{H}^{(n)}\, d\tau d\tau',$$

whence it follows that
$$J_{mn} = J_{mn}^{\dagger} = \omega^2\mu_0 D_{mn}, \tag{25}$$

or
$$J_{mn} = J_{mn}^{\dagger} = \frac{J_{mn}\, J_{mn}^{\dagger}}{\omega^2\mu_0\, D_{mn}}, \tag{26}$$

where

$$D_{nm} = \int \mathbf{E}^{(n)\dagger}.\boldsymbol{\epsilon}'.\mathbf{E}^{(m)}\,d\tau + \int \mathbf{H}^{(n)\dagger}.\boldsymbol{\mu}'.\mathbf{H}^{(m)}\,d\tau -$$
$$-\omega^2\mu_0 \iint \mathbf{E}^{(n)\dagger}.\boldsymbol{\epsilon}'.\mathbf{N}^T(\mathbf{r}'|\mathbf{r}).\boldsymbol{\epsilon}'.\mathbf{E}^{(m)}\,d\tau d\tau' -$$
$$-\omega^2\epsilon_0 \iint \mathbf{H}^{(n)\dagger}.\boldsymbol{\mu}'.\mathbf{M}^T(\mathbf{r}'|\mathbf{r}).\boldsymbol{\mu}'.\mathbf{H}^{(m)}\,d\tau d\tau' +$$
$$+j\omega \iint \mathbf{E}^{(n)\dagger}.\boldsymbol{\epsilon}'.[\boldsymbol{\nabla}\times\mathbf{M}(\mathbf{r}|\mathbf{r}')].\boldsymbol{\mu}'.\mathbf{H}^{(m)}\,d\tau d\tau' -$$
$$-j\omega \iint \mathbf{H}^{(n)\dagger}.\boldsymbol{\mu}'.[\boldsymbol{\nabla}\times\mathbf{N}(\mathbf{r}|\mathbf{r}')].\boldsymbol{\epsilon}'.\mathbf{E}^{(m)}\,d\tau d\tau'. \qquad (27)$$

It is readily shown that the first-order variation of the right-hand side of equation (26) with respect to the fields, subject to condition (25), yields the integral equations for the electromagnetic fields within the obstacle. Since expression (26) is amplitude independent, condition (25) is automatically satisfied. Equation (26) is therefore a variational expression for the approximate evaluation of the elements J_{nm}.

4. Application of the variational principle

The actual application of the variational principle to practical examples will be presented in a forthcoming paper. In this section we shall indicate some of the different ways in which equation (26) can be utilized in order to obtain practical approximate results for the reflection and transmission coefficients of an obstacle.

The full use of the variational method consists of the substitution of trial functions with unknown variational parameters into the variational expression (26), and the subsequent determination of the values of these parameters which make the variational expression an extremum. This may seem like a tedious task, and the usefulness of the variational method thus be questioned. However, unless one is interested in obtaining only first-order results for very small obstacles, this may be the only feasible, in any case the most fruitful approach.

If one is interested in obtaining only first-order results, one may do so by simply substituting completely determined trial functions into equation (26). Alternatively, equation (20) may be used, choosing a trial field which satisfies equations (25). Since D_{nm} is proportional to the square and J_{nm} to the first power of the amplitude of the field, we can always find an amplitude which will make the trial field satisfy equations (25). For a given trial field, the one whose amplitude is so adjusted will yield the most accurate results.

Equations (20) are exact formulae, which have been used by Nikol'skii (5) to compute the phase shifts produced by small (obstacle dimension

$\ll$ wavelength λ) gyrotropic obstacles in waveguides utilizing trial functions which he obtains from a quasi-stationary approximation. This approximation restricts his results to lossless media (5) and makes his result dependent upon the amplitude of the trial field.

A seemingly different approach has been employed by Berk and Epstein (6) to obtain first-order results for the problem of a ferrite post in a rectangular waveguide (Fig. 1). Their result, however, is another example

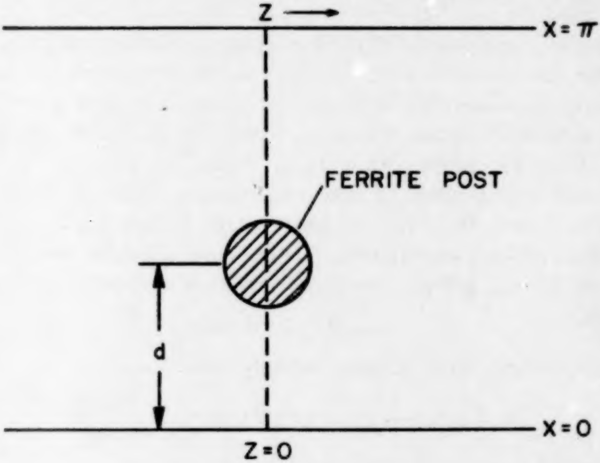

FIG. 1

of the application of equations (20) with a slight modification. This modification involves changing the volume integrals in equations (20) to surface integrals.

The wave equation satisfied by the propagating electric field

$$\mathbf{E}_p^{(i)} = \mathbf{E}_1^{(i)} e^{(-1)^i j\gamma_1 z}$$

is

$$\nabla\times\nabla\times\mathbf{E}_p^{(i)} - k^2\mathbf{E}_p^{(i)} = 0.$$

Multiplying this equation by $\mathbf{E}^{(j)}$, equation (2a) by $\mathbf{E}_p^{(i)}$, and subtracting, we find

$$\begin{aligned}\nabla.[\mathbf{E}^{(j)}\times(\nabla\times\mathbf{E}_p^{(i)}) - \mathbf{E}_p^{(i)}\times(\nabla\times\mathbf{E}^{(j)})] \\ &= \omega^2\mu_0\mathbf{E}_p^{(i)}.\boldsymbol{\epsilon}'.\mathbf{E}^{(j)} - j\omega\mathbf{E}_p^{(i)}.\nabla\times(\boldsymbol{\mu}'.\mathbf{H}^{(j)}) \\ &= \omega^2\mu_0\mathbf{E}_p^{(i)}.\boldsymbol{\epsilon}'.\mathbf{E}^{(j)} - j\omega(\nabla\times\mathbf{E}_p^{(i)}).(\boldsymbol{\mu}'.\mathbf{H}^{(j)}) + \\ &\qquad + j\omega\nabla.[\mathbf{E}_p^{(i)}\times(\boldsymbol{\mu}'.\mathbf{H}^{(j)})].\end{aligned}$$

Integrating over the volume of the obstacle and setting

$$\nabla\times\mathbf{E}_p^{(i)} = -j\omega\mu_0\mathbf{H}_p^{(i)}$$

yields with the help of the divergence theorem

$$\omega^2\mu_0\Big[\int \mathbf{E}_P^{(i)}.\boldsymbol{\epsilon}'.\mathbf{E}^{(j)}\,d\tau - \int \mathbf{H}_P^{(i)}.(\boldsymbol{\mu}'.\mathbf{H}^{(j)})\,d\tau\Big]$$
$$= -j\omega\mu_0\int[\mathbf{E}^{(j)}\times\mathbf{H}_P^{(i)}-\mathbf{E}_P^{(i)}\times\mathbf{H}^{(j)}].\mathbf{n}\,dS. \quad (28)$$

One may thus evaluate the reflection and transmission coefficients from the tangential components of $\mathbf{E}^{(j)}$ and $\mathbf{H}^{(j)}$ on the boundary of the obstacle by using equations (20) as perturbation formulae. Berk and Epstein used for their approximate field, the long wavelength limit of the tangential components of the free space scattering problem.

To show that the surface integrals appearing in equation (28) are indeed the same as the ones used by Berk and Epstein, we note that Berk and Epstein restricted themselves to the problem of the scattering of the lowest TE mode, having an electric vector parallel to the rod axis (y-axis) and independent of the y-coordinate. The symmetry of the problem indicates that $\mathbf{E}^{(j)}$ likewise points in the y-direction and is independent of the y-coordinate. Furthermore, since the tangential components of $\mathbf{E}^{(j)}$ and $\mathbf{H}^{(j)}$ are continuous, we may utilize the external fields for which

$$-j\omega\mu_0\mathbf{H}_{\text{ext}}^{(j)} = \nabla\times\mathbf{E}_{\text{ext}}^{(j)}.$$

For y-independent fields pointing in the y-direction

$$-j\omega\mu_0(\mathbf{n}\times\mathbf{H}_{\text{ext}}^{(j)}) = \mathbf{n}\times\nabla\times\mathbf{E}_{\text{ext}}^{(j)} = -\frac{\partial\mathbf{E}_{\text{ext}}^{(j)}}{\partial n}$$

on the curved boundary. Thus

$$J_{ij} = \int\left(\mathscr{E}_{\text{ext}}^{(j)}\frac{\partial\mathscr{E}_P^{(i)}}{\partial n}-\mathscr{E}_P^{(i)}\frac{\partial\mathscr{E}_{\text{ext}}^{(j)}}{\partial n}\right)dS', \quad (29)$$

where $\mathscr{E}$ is the magnitude of $\mathbf{E}$, and the integration is performed only over the curved boundary of the rod as $\mathbf{n}\times\mathbf{E}$ is zero at the guide boundary. If like Berk and Epstein we now set

$$\mathscr{E}^{(j)} = \mathscr{E}_P^{(j)}+\mathscr{E}_s^{(j)},$$

we obtain
$$J_{ij} = \int\left(\mathscr{E}_s^{(j)}\frac{\partial\mathscr{E}_P^{(i)}}{\partial r}-\mathscr{E}_P^{(i)}\frac{\partial\mathscr{E}_s^{(j)}}{\partial r}\right)ds\,dy \quad (30)$$

as
$$\int\left(\mathscr{E}_P^{(i)}\frac{\partial\mathscr{E}_P^{(j)}}{\partial n}-\frac{\partial\mathscr{E}_P^{(i)}}{\partial n}\mathscr{E}_P^{(j)}\right)ds\,dy = 0.$$

The y-integration may be omitted if it is likewise omitted in the evaluation of A_1. Formula (30) is the result one obtains from formula (6.6) of Berk and Epstein if only the contribution from the first term is used in the eigenfunction expansion of the Green's function, as was done by Berk and Epstein in their section 8. Again we note that the results of Berk

and Epstein are restricted to very thin rods not too close to the guide wall and are dependent upon the amplitude of the trial field.

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A HELICAL COORDINATE SYSTEM AND ITS APPLICATIONS IN ELECTROMAGNETIC THEORY

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[Received 26 September 1957]

SUMMARY

A helical coordinate system is described and its elementary properties discussed. Several electromagnetic problems are solved as illustrations of the use of the system. Its advantages are twofold: firstly, it enables problems involving helical symmetry to be solved exactly, and secondly, it shows in a new light problems which have previously been solved in cylindrical coordinates. The closing section discusses points arising out of the examples treated; in particular it is found that the characteristic equation for wave propagation in a helical system is always identical with that for an appropriate cylindrical analogue, if the usual condition that fields must repeat at intervals of 2π in the azimuthal dimension is removed.

1. List of principal symbols

THE following list defines, with their usual meanings, the symbols most commonly used throughout the paper. Other symbols which occur only occasionally, or any of the following which are used with meanings different from those given below, are defined in the text where they occur. Where the meaning of a symbol is obvious it is not defined.

r, θ, z	cylindrical coordinates.
r, ϕ, ζ	helical coordinates.
p	pitch of the helical coordinate system.
α	pitch angle of the helical coordinate system, defined by $\tan\alpha = p/2\pi r$, $0 \leqslant \alpha < \frac{1}{2}\pi$.
β	parameter giving the z or ζ dependence of electromagnetic fields.
n	parameter giving the ϕ dependence of fields.
q	parameter giving the θ dependence of fields.
ν	an integer giving the ζ dependence in certain cases.
a	radius of a cylindrical surface; when two are present, the outer one.
b	radius of helical wire or an inner cylindrical surface.
c	ζ-dimension of helical waveguide, and length of air space in helically supported coaxial line.

E	electric field.
H	magnetic field.
V	arbitrary vector.
ψ	arbitrary scalar.
ϵ_0, μ_0	permittivity and permeability, respectively, of free space.
λ_0	free-space wavelength.
$k^2 =$	$\omega^2\epsilon_0\mu_0-\beta^2$.
$X_q(x)$	linear function of the Bessel functions $J_q(x)$ and $Y_q(x)$.
$X'_q(x) = \frac{dX_q}{dx}$.	

2. Introduction

The helical coordinate system treated in this paper, and which the author believes has not been previously discussed, is simply related to the cylindrical polar system, and may in fact be applied to any problem to which cylindrical polar coordinates are appropriate. An example of this is the cylindrical waveguide, which is normally treated in cylindrical coordinates. The so-called normal modes are in fact degenerate, each being the result of a superposition of two helical modes. This is, of course, well known, but the helical modes are usually spoken of as 'circularly polarized'. It will be clear from the treatment given below (section 4.2) that a more proper term would be 'helically polarized'.

An example of a problem to which the present helical coordinate system is not suited is the twisted waveguide, which Lewin has treated in a helical system related to the Cartesian system (**1**). It might therefore help to distinguish the two systems if we call Lewin's coordinates the 'rectangular helical system' and those discussed in the present paper the 'polar helical system'. We shall, however, not refer again to Lewin's system in this paper, so there will be no confusion if we drop the term 'polar'.

We shall first give the fundamental theory of the helical coordinate system, then solve several electromagnetic problems in the system, and suggest other problems, not only electromagnetic, to which it may be appropriate. Finally, we shall discuss various points arising out of the examples treated here and prove a theorem which enables a great simplification to be made in a helical problem by reducing it to an equivalent cylindrical one.

3. The helical coordinate system

3.1. *Description of the system*

Consider first a set of cylindrical coordinates r, θ, z (Fig. 1). The coordinate surfaces, which are orthogonal, are a set of planes intersecting in a

common axis, the z-axis, a set of parallel planes, orthogonal to the first set, and a set of concentric cylinders with the z-axis as axis. These families of surfaces are given respectively by $\theta =$ constant, $z =$ constant, and $r =$ constant.

In the helical system we shall retain the family of cylinders $r =$ constant, with meaning unchanged. In place of the parallel planes $z =$ constant, we shall take a family of helical surfaces given by $z = \text{constant} + p\theta/2\pi$, p

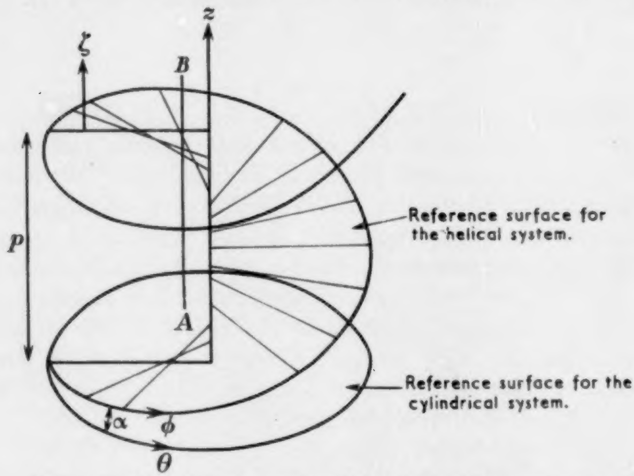

FIG. 1. Helical reference surface for right-handed coordinates.

being the pitch; the surface $z = p\theta/2\pi$ is illustrated in Fig. 1. The coordinate z in the cylindrical system is replaced by $\zeta = z - p\theta/2\pi$, i.e. ζ is measured parallel to z, from the surface $z = p\theta/2\pi$, which in the helical system becomes $\zeta = 0$. Our third set of coordinate surfaces is the set of planes $\theta =$ constant; however, we shall no longer use θ, but ϕ. ϕ is numerically equal to θ, but whereas θ is measured in a plane $z =$ constant, ϕ is measured in a helical surface $\zeta =$ constant. It will be seen that this helical system is not orthogonal.

One further quantity that will be useful is the pitch angle α, which is the angle made with a plane $z =$ constant by the tangent to the curve $\zeta =$ constant, $r =$ constant. Its value is given by equation (3) below.

It might be thought that it would be more convenient to use the system consisting of the surfaces $r =$ constant, and $\phi =$ constant, described above, together with a third set of helical surfaces $\psi =$ constant, the ψ surfaces being orthogonal to the ϕ and r surfaces, and left-handed if the ϕ surfaces are right-handed. However, the analysis in such a system is very difficult;

in particular the wave equation is complicated by the fact that the resolutes of $\nabla^2 V$ are not the same as those of the operator (div grad)V.

3.2. *Helical spaces*

Geometrically, we may reach a point $(r, \theta, z+p)$ from the point (r, θ, z) in one of two ways; we may travel along the straight line $r = r$, $\phi = \theta$, i.e. suffer a translation in the ζ-direction; or we may travel along the helical path $r = r$, $\zeta =$ constant, i.e. suffer a translation in ϕ. These two geometrically equivalent changes of position may or may not be physically equivalent, according to what materials may be present, in what positions, in the system under consideration. We may think of the equivalence or non-equivalence of the translations as dependent not on the materials present but on the properties of the space we are dealing with. This enables us to express boundary conditions in a convenient way; according to the problem under consideration, we choose the appropriate type of space, and the geometry then takes care of itself.

We shall now define our types of space. If the two translations described above are equivalent, we shall describe the space as multiply connected. If the translations are not equivalent, we shall describe the space as simply connected. There is a third type of space; its definition will be better understood after some discussion of simply and multiply connected spaces.

In simply connected space, the helical surface $\zeta = 0$ constitutes a barrier which cannot be crossed; it is impossible to reach B from A (Fig. 1) by travelling parallel to the z-axis, but it can be reached by varying ϕ. A point can be expressed in one way only, by suitable values of r and ϕ, and a value of ζ lying between 0 and p.

In multiply connected space, the helical surface $\zeta = 0$ is not a barrier, and B can be reached from A by changing ζ an amount p as well as by changing ϕ by 2π. If A is the point (r, ϕ, ζ), B may be represented by $(r, \phi+2\pi, \zeta)$ or by $(r, \phi, \zeta+p)$. Thus any point can be represented in an infinite number of ways by a unique value of r and a value of ζ which depends on the value assigned to ϕ.

The regions between the surfaces $\zeta = np$ and $\zeta = (n+1)p$, for all n, are contiguous in multiply connected space; in simply connected space they are not, so that by crossing the surfaces $\zeta = 0$ or $\zeta = p$, we go outside the system. In multiply connected space, when we cross the surfaces $\zeta = 0$ or $\zeta = p$, we move to another part of the system.

The third type of space is that in which ζ is not limited to the range $0 \leqslant \zeta \leqslant p$, but a change of p in ζ is not equivalent to a change of 2π in ϕ. This type of space may be called disconnected space. It is probably of no physical interest, and will not be considered further in this paper.

We shall treat below several electromagnetic problems; the coordinate surfaces will be so chosen that surfaces of dielectric and conducting media are expressible as a single coordinate equated to a constant, or so as to coincide with equipotential surfaces. We shall postpone further discussion of types of space till section 6, since the arguments will be easier to follow in the light of the examples.

Left- and right-handedness

When referring in future to cylindrical coordinates, these will be taken to be right-handed—that is, on looking along the z-axis in the positive direction, θ is measured in the clockwise sense. We shall measure ζ and ϕ in the same sense as z and θ. If the helical surfaces are right-handed, as in Fig. 1, we shall, in making a positive change in ϕ with constant ζ, make a positive change in z, so that the pitch p is positive. Similarly, if the helical coordinate surfaces are left-handed, p is negative.

3.3. *Elementary geometrical properties*

The following formulae may be derived fairly easily by elementary methods, and we state them here without proof.

Coordinates

Although these have been fully described in section 3.1, we give the equations again here for reference. They are, for the helical system in terms of the cylindrical system,

$$\left.\begin{array}{c} r = r \\ \phi = \theta \text{ in magnitude,} \\ \text{but is measured along a different curve, namely,} \\ \zeta = z - \dfrac{p\theta}{2\pi} \end{array}\right\}. \tag{1}$$

For the cylindrical system in terms of the helical system, we have

$$z = \zeta + \frac{p\phi}{2\pi}. \tag{2}$$

We also have the pitch angle α, given by

$$\tan\alpha = \frac{p}{2\pi r} \quad (0 \leqslant \alpha < \tfrac{1}{2}\pi). \tag{3}$$

Line elements

$$\left.\begin{aligned} ds_1 &= dr \\ ds_2 &= \sqrt{(r^2 + p^2/4\pi^2)}\, d\phi \\ ds_3 &= d\zeta \end{aligned}\right\}. \tag{4}$$

Surface elements

$$\left.\begin{aligned} dS_1 &= \mathbf{m}.\mathbf{n}\sqrt{(r^2+p^2/4\pi^2)}\,d\phi d\zeta = r\,d\phi d\zeta \\ dS_2 &= drd\zeta \\ dS_3 &= \sqrt{(r^2+p^2/4\pi^2)}\,d\phi dr \end{aligned}\right\}. \tag{5}$$

Here **m** and **n** are unit vectors parallel to ds_2 and ds_3 respectively.

Volume element $$dV = r\,drd\phi d\zeta. \tag{6}$$

Distance element

$$ds^2 = dr^2+(r^2+p^2/4\pi^2)\,d\phi^2+d\zeta^2+2(p/2\pi)\,d\phi d\zeta. \tag{7}$$

Differential relations

$$\left.\begin{aligned} &\frac{\partial\zeta}{\partial z} = \frac{\partial z}{\partial\zeta} = 1 \\ &\frac{\partial}{\partial\zeta} = \frac{\partial}{\partial z} \\ &\frac{\partial\phi}{\partial\theta} = \frac{\partial\theta}{\partial\phi} = 1 \\ &\frac{1}{r}\frac{\partial}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial\phi}-\frac{p}{2\pi r}\frac{\partial}{\partial\zeta} \\ &\frac{1}{r}\frac{\partial}{\partial\phi} = \frac{1}{r}\frac{\partial}{\partial\theta}+\frac{p}{2\pi r}\frac{\partial}{\partial z} \\ &\frac{\partial\phi}{\partial\zeta} = \frac{\partial\zeta}{\partial\phi} = \frac{\partial\theta}{\partial z} = \frac{\partial z}{\partial\theta} = \frac{\partial\theta}{\partial\zeta} = \frac{\partial\phi}{\partial z} = 0 \\ &\frac{\partial z}{\partial\phi} = \frac{p}{2\pi} \\ &\frac{\partial\zeta}{\partial\theta} = -\frac{p}{2\pi} \end{aligned}\right\}. \tag{8}$$

3.4. *Vector relations*

Vector components

Since the radial component of a vector is perpendicular to both θ and z and to ϕ and ζ, it will be the same in helical coordinates as in cylindrical coordinates. Let us consider now a vector **V** having components $\mathbf{V}_\theta$, $\mathbf{V}_z$, and confine our attention to a surface $r =$ constant. The components of **V** in both the helical and cylindrical systems are illustrated in Fig. 2. It may readily be seen that

$$\left.\begin{aligned} V_\phi &= V_\theta\sec\alpha; & V_\theta &= V_\phi\cos\alpha \\ V_\zeta &= V_z-V_\theta\tan\alpha; & V_z &= V_\zeta+V_\phi\sin\alpha \end{aligned}\right\}. \tag{9}$$

Gradient

In a cylindrical coordinate system, we have

$$(\operatorname{grad}\psi)_r = \frac{\partial\psi}{\partial r},$$

$$(\operatorname{grad}\psi)_\theta = \frac{1}{r}\frac{\partial\psi}{\partial\theta},$$

$$(\operatorname{grad}\psi)_z = \frac{\partial\psi}{\partial z}.$$

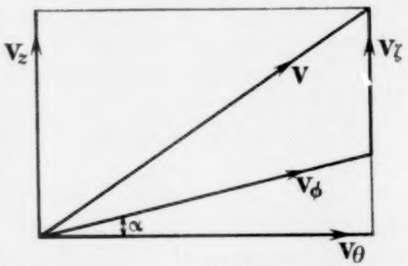

FIG. 2. Vector components in a surface of constant r.

If we let $\operatorname{grad}\psi$ be the vector $\mathbf{V}$, we can calculate its value in helical coordinates by the use of the relations (9). We thus have

$$(\operatorname{grad}\psi)_r = \frac{\partial\psi}{\partial r},$$

$$(\operatorname{grad}\psi)_\phi = \frac{1}{r\cos\alpha}\frac{\partial\psi}{\partial\theta},$$

$$(\operatorname{grad}\psi) = \frac{\partial\psi}{\partial z} - \frac{\tan\alpha}{r}\frac{\partial\psi}{\partial\theta}.$$

Substituting for $\partial\psi/\partial\theta$ and $\partial\psi/\partial z$ from equations (8), we obtain

$$\operatorname{grad}\psi = \mathbf{l}\frac{\partial\psi}{\partial r} + \frac{\mathbf{m}}{\cos\alpha}\left[\frac{1}{r}\frac{\partial\psi}{\partial\phi} - \frac{p}{2\pi r}\frac{\partial\psi}{\partial\zeta}\right] + \mathbf{n}\left[\frac{\partial\psi}{\partial\zeta} - \frac{\tan\alpha}{r}\frac{\partial\psi}{\partial\phi} + \tan^2\alpha\frac{\partial\psi}{\partial\zeta}\right], \quad (10)$$

where $\mathbf{l}$, $\mathbf{m}$, and $\mathbf{n}$ are unit vectors parallel to ds_1, ds_2, ds_3.

Divergence

We can write

$$\nabla.\mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r) + (\nabla_\phi + \nabla_\zeta)(\mathbf{V}_\phi + \mathbf{V}_\zeta).$$

∇_ϕ and ∇_ζ can be obtained from equation (10); it is important to remember

in forming the scalar product that $\nabla_\phi . \mathbf{V}_\zeta$ and $\nabla_\zeta . \mathbf{V}_\phi$ are not zero, since the ϕ and ζ coordinates are not orthogonal. We thus obtain

$$\operatorname{div} \mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(rV_r)+\frac{\cos\alpha}{r}\frac{\partial V_\phi}{\partial \phi}+\frac{\partial V_\zeta}{\partial \zeta} \tag{11}$$

the cross terms having cancelled out.

Curl

The simplest way to treat the curl of a vector $\mathbf{V}$ is to write down the expression for curl $\mathbf{V}$ in cylindrical coordinates and then transform to helical coordinates by means of the relations (8), simultaneously replacing the cylindrical components of $\mathbf{V}$ by helical components according to equations (9). The results are

$$\left.\begin{aligned}
\operatorname{curl}_r \mathbf{V} &= \left(\frac{1}{r}\frac{\partial V_\zeta}{\partial \phi}-\frac{p}{2\pi r}\frac{\partial V_\zeta}{\partial \zeta}\right)+\left(\frac{\sin\alpha}{r}\frac{\partial V_\phi}{\partial \phi}-\frac{1}{\cos\alpha}\frac{\partial V_\phi}{\partial \zeta}\right)\\
\operatorname{curl}_\phi \mathbf{V} &= \frac{1}{\cos\alpha}\left\{\frac{\partial V_r}{\partial \zeta}-\frac{\partial V_\zeta}{\partial r}-\frac{\partial}{\partial r}(V_\phi \sin\alpha)\right\}\\
\operatorname{curl}_\zeta \mathbf{V} &= \left\{\frac{1}{\sqrt{(r^2+p^2/4\pi^2)}}\frac{\partial}{\partial r}(rV_\phi)+\frac{\sin^2\alpha}{\cos\alpha}\frac{\partial V_\phi}{\partial r}\right\}-\left\{\frac{1}{r}\frac{\partial V_r}{\partial \phi}-\frac{p}{2\pi r}\frac{\partial V_\zeta}{\partial r}\right\}
\end{aligned}\right\}. \tag{12}$$

3.5. *The Laplacian operator*

If in equation (11) we replace the vector $\mathbf{V}$ by grad ψ as given by equation (10), we obtain

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial \phi^2}+\frac{1}{\cos^2\alpha}\frac{\partial^2\psi}{\partial \zeta^2}-\frac{2}{r}\frac{\sin\alpha}{\cos\alpha}\frac{\partial^2\psi}{\partial \zeta\partial \phi} \tag{13}$$

which may be written

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\psi}{\partial \phi^2}+\left(1+\frac{p^2}{4\pi^2r^2}\right)\frac{\partial^2\psi}{\partial \zeta^2}-2\left(\frac{p}{2\pi r^2}\right)\frac{\partial^2\psi}{\partial \zeta\partial \phi}. \tag{14}$$

The symmetry of a physical system will often impose the condition $\partial/\partial\phi = 0$, i.e. the helical coordinate surfaces may be chosen to be equipotential surfaces. In this case, the Laplacian reduces to the simpler form

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right)+\left(1+\frac{p^2}{4\pi^2r^2}\right)\frac{\partial^2\psi}{\partial \zeta^2}. \tag{15}$$

The wave equation

We have

$$\left(\nabla^2-\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\right)\psi = 0, \tag{16}$$

where v is the velocity of the waves. Let us assume a time variation of the form $e^{j\omega t}$, and write $\omega^2/v^2 = \chi^2$. The wave equation then becomes

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}+\left(1+\frac{p^2}{4\pi^2r^2}\right)\frac{\partial^2\psi}{\partial\zeta^2}-2\left(\frac{p}{2\pi r^2}\right)\frac{\partial^2\psi}{\partial\zeta\partial\phi}+\chi^2\psi = 0 \tag{17}$$

and we shall look for solutions of the form

$$\psi = R(r)P(\phi)Z(\zeta). \tag{18}$$

Substituting for ψ from equation (18) into equation (17), we obtain

$$\frac{R''}{R}+\frac{R'}{rR}+\frac{P''}{r^2P}+\left(1+\frac{p^2}{4\pi^2r^2}\right)\frac{Z''}{Z}-\frac{pP'Z'}{\pi r^2PZ}+\chi^2 = 0, \tag{19}$$

where the primes denote differentiation with respect to r, ϕ, or ζ as the case may be. We cannot now separate the variables, but we can obtain an equation in r alone if P''/P, $P'Z'/PZ$, and Z''/Z are all constant. By analogy with the case of cylindrical coordinates, we assume

$$\left.\begin{aligned} Z &= e^{\pm j\beta\zeta} \\ P &= e^{\pm jn\phi} \end{aligned}\right\} \tag{20}$$

which gives

$$\left.\begin{aligned} \frac{Z''}{Z} &= -\beta^2 \\ \frac{P''}{P} &= -n^2 \\ \frac{P'Z'}{PZ} &= -n\beta \end{aligned}\right\} \tag{21}$$

and equation (19) then reduces to

$$\frac{R''}{R}+\frac{R'}{rR}+k^2-\frac{q^2}{r^2} = 0, \tag{22}$$

where

$$q = n-\frac{p\beta}{2\pi} \tag{23}$$

and

$$k^2 = \chi^2-\beta^2. \tag{24}$$

Equation (22) becomes Bessel's equation of order q if we take kr as the independent variable. The solution is thus

$$R = X_q(kr), \tag{25}$$

where $X_q(x)$ denotes a linear combination of $J_q(x)$ and $Y_q(x)$.

There is always the possibility, of course, that other solutions of the wave equation exist which are not obtained in this way, but this applies equally to all coordinate systems, since the procedure we have followed here is the one commonly used in other coordinate systems.

The solution of the wave equation in helical coordinates is thus seen to

be related to its solution in cylindrical coordinates. The differences are that the order of the Bessel function is not the same as the n in the ϕ dependence, and n and q are not constrained, in general, to be integers. It will often be convenient to choose the helical coordinate surfaces ($\zeta =$ constant) to be equipotential surfaces; in this case, $\partial/\partial\phi = 0$, i.e. $n = 0$ and $q = -p\beta/2\pi$.

It should be noted that in a right-handed system we must take the $-$ signs in both equations (20) for a forward wave, or the $+$ signs in both equations for a backward wave. We have also pointed out above that p is positive. In a left-handed system, we must take one $+$ sign and one $-$ sign in equations (20), and p is negative. Thus $p\beta$ and n have the same sign in either system for forward waves and for backward waves. In changing from a wave in one direction to a wave in the other, or from a system of one hand to a system of the other hand, q changes sign, but retains the same numerical value, and R is unchanged, as we should expect. The situation may be summed up thus:

	Forward wave		*Backward wave*	
	Right-handed coordinates	*Left-handed coordinates*	*Right-handed coordinates*	*Left-handed coordinates*
p	+ive	−ive	+ive	−ive
β	−ive	−ive	+ive	+ive
n	−ive	+ive	+ive	−ive
q	+ive or −ive	−ive or +ive	−ive or +ive	+ive or −ive

The sign of q will depend on the relative magnitudes of n and $p\beta/2\pi$. If n is less than $p\beta/2\pi$, q is to have the sign given first in each column of the table. This will usually be the case, and is so in particular when $n = 0$.

4. Electromagnetic theory

4.1. *Fundamentals*

In section 3.5 we have solved the wave equation for an undefined scalar independent variable ψ. In electromagnetic theory this variable will be the magnitude of an electric or magnetic field component, and in cylindrical coordinates it would be E_z or H_z. In helical coordinates the variable is $E_\zeta+E_\phi \sin\alpha$ or $H_\zeta+H_\phi \sin\alpha$. One of these can be put equal to 0 in most cases to give H-waves or E-waves. We shall also have $\chi^2 = \omega^2\epsilon_0\mu_0$. By virtue of equations (20), (23), (24), and (25), we have, for forward E-waves in a right-handed system,

$$\left.\begin{aligned} E_\zeta+E_\phi \sin\alpha &= X_q(kr)e^{-j\beta\zeta}e^{-jn\phi}e^{j\omega t} \\ H_\zeta+H_\phi \sin\alpha &= 0 \end{aligned}\right\} \tag{26}$$

while for H-waves

$$\left.\begin{aligned} E_\zeta + E_\phi \sin\alpha &= 0 \\ H_\zeta + H_\phi \sin\alpha &= X_q(kr)e^{-j\beta\zeta}e^{-jn\phi}e^{j\omega t} \end{aligned}\right\}. \tag{27}$$

It is possible to express all the field components in terms of E_z and H_z by separating them out in turn from Maxwell's equations. This, however, is a tedious task in helical coordinates, and it is simpler to take the expressions for the field components in cylindrical coordinates and then convert to helical coordinates with the aid of equations (8) and (9). For a system *in vacuo* we obtain

$$\left.\begin{aligned} E_r &= \frac{1}{k^2}\left\{\frac{\omega\mu_0 q}{r}[H_\zeta + H_\phi \sin\alpha] + j\beta\left[\frac{\partial E_\zeta}{\partial r} + \frac{\partial}{\partial r}(E_\phi \sin\alpha)\right]\right\} \\ E_\phi &= \frac{1}{k^2\cos\alpha}\left\{j\omega\mu_0\left[\frac{\partial H_\zeta}{\partial r} + \frac{\partial}{\partial r}(H_\phi \sin\alpha)\right] - \frac{\beta q}{r}[E_\zeta + E_\phi \sin\alpha]\right\} \\ H_r &= \frac{1}{k^2}\left\{j\beta\left[\frac{\partial H_\zeta}{\partial r} + \frac{\partial}{\partial r}(H_\phi \sin\alpha)\right] - \frac{\omega\epsilon_0 q}{r}[E_\zeta + E_\phi \sin\alpha]\right\} \\ H_\phi &= \frac{1}{k^2\cos\alpha}\left\{\frac{-\beta q}{r}[H_\zeta + H_\phi \sin\alpha] - j\omega\epsilon_0\left[\frac{\partial E_\zeta}{\partial r} + \frac{\partial}{\partial r}(E_\phi \sin\alpha)\right]\right\} \end{aligned}\right\}. \tag{28}$$

Substituting from equations (26) and (27) into equation (28), we obtain for forward E-waves in a right-handed system

$$\left.\begin{aligned} E_r &= \frac{j\beta}{k} X_q'(kr)e^{-j\beta\zeta}e^{-jn\phi} \\ E_\phi &= \frac{-\beta q}{k^2 r\cos\alpha} X_q(kr)e^{-j\beta\zeta}e^{-jn\phi} \\ E_\zeta &= \left(1 + \frac{p\beta q}{2\pi k^2 r^2}\right) X_q(kr)e^{-j\beta\zeta}e^{-jn\phi} \\ H_r &= \frac{-\omega\epsilon_0 q}{k^2 r} X_q(kr)e^{-j\beta\zeta}e^{-jn\phi} \\ H_\phi &= \frac{-j\omega\epsilon_0}{k\cos\alpha} X_q'(kr)e^{-j\beta\zeta}e^{-jn\phi} \\ H_\zeta &= \frac{j\omega\epsilon_0 p}{2\pi kr} X_q'(kr)e^{-j\beta\zeta}e^{-jn\phi} \end{aligned}\right\}, \tag{29}$$

where $X'(kr) = dX_q/d(kr)$, and the factor $e^{j\omega t}$ is understood; for H-waves we have

$$\left.\begin{aligned} E_r &= \frac{\omega\mu_0 q}{k^2 r} X_q(kr)e^{-j\beta\zeta}e^{-jn\phi} \\ E_\phi &= \frac{j\omega\mu_0}{k\cos\alpha} X_q'(kr)e^{-j\beta\zeta}e^{-jn\phi} \\ E_\zeta &= \frac{-j\omega\mu_0 p}{2\pi kr} X_q'(kr)e^{-j\beta\zeta}e^{-jn\phi} \end{aligned}\right\}. \tag{30}$$

$$\left.\begin{aligned} H_r &= \frac{j\beta}{k} X'_q(kr)e^{-j\beta\zeta}e^{-jn\phi} \\ H_\phi &= \frac{-\beta q}{k^2 r\cos\alpha} X_q(kr)e^{-j\beta\zeta}e^{-jn\phi} \\ H_\zeta &= \left(1+\frac{p\beta q}{2\pi k^2 r^2}\right) X_q(kr)e^{-j\beta\zeta}e^{-jn\phi} \end{aligned}\right\}$$

For a left-handed system, replace p by $-p$ and n by $-n$ in equations (29) and (30). For backward waves, replace β by $-\beta$ and n by $-n$. For backward waves in a left-handed system, make both these changes; the net effect is to replace p and β by $-p$ and $-\beta$ respectively, the two changes in n having cancelled. In making these changes, it must be remembered that q is a function of p, β, and n. Equations (29) and (30) apply equally to a system filled with a homogeneous isotropic dielectric or magnetic material, if ϵ_0 and μ_0 are replaced by $\epsilon\epsilon_0$ or $\mu\mu_0$.

We shall now treat various examples; in the main, the problems amount to substituting the appropriate boundary conditions into equations (29) and (30) and so deriving a characteristic equation. However, it is not always β that the characteristic equation is to be solved for.

4.2. *The cylindrical waveguide*

Since the system contains the axis $r = 0$, at which the fields must remain finite, the solution of the wave equation that involves $Y_q(kr)$ is excluded, so that the field equations are as given in equations (29) and (30) with $X_q(kr)$ replaced by $J_q(kr)$. The helical coordinate surfaces are conveniently chosen to be equipotential surfaces, so that $n = 0$ and p is thus equal to q wavelengths. Now

$$e^{-j\beta\zeta} = e^{-j\beta z}e^{-jp\beta\theta/2\pi} = e^{-j\beta z}e^{-jq\theta},$$

so q must be an integer and p is therefore an integral number of wavelengths.

We now apply the boundary condition that $E_\phi = 0$ at $r = a$, a being the radius of the waveguide. From equations (29) and (30) we thus obtain

$$J_q(ka) = 0 \tag{31}$$

for E-waves, and

$$J'_q(ka) = 0 \tag{32}$$

for H-waves. These are the characteristic equations, to be solved for k and hence β. They are identical with the characteristic equations obtained by the treatment in cylindrical coordinates, but do not refer to the same kinds of waves. In cylindrical coordinates, we should have been dealing with waves in which a given point in the wavefront moves parallel to the axis of the guide; in the present case, a given point in the wave front

moves along a helical path. Such waves are commonly said to be circularly polarized; it is clear, however, that a better term would be 'helically polarized'. Circular polarization only exists in an infinite homogeneous medium.

In a cylindrical waveguide, there is no reason why one sense of helical polarization should be preferred to the other. Let us therefore consider the case of two oppositely helically polarized waves, of the same frequency, having equal amplitudes, travelling together in the same direction along a guide. We denote by S any one of the field components, for an E- or H-wave, and write for the right-handed wave

$$S_+ = Kf(r)e^{-j\beta\zeta},$$

which may be written

$$S_+ = Kf(r)e^{-j\beta z}e^{-jq\theta}.$$

For the left-handed wave,

$$S_- = \pm Kf(r)e^{-j\beta z}e^{jq\theta},$$

where the $+$ or $-$ sign is to be chosen according to the values of the constant K and of the radial function f and the parity of q. For the composite wave $S_+ + S_-$, we obtain

$$S = Kf(r)e^{-j\beta z}(e^{jq\theta} \pm e^{-jq\theta}),$$

i.e.

$$S = Kf(r)e^{-j\beta z}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}q\theta, \tag{33}$$

which is identical with the field component found by the treatment in cylindrical coordinates. This demonstrates the degeneracy of the normal modes of a cylindrical waveguide, each of which is a combination of two helically-polarized waves. Apart from this aspect, the example is, perhaps, trivial; it does serve, however, to give us some feeling for the helical coordinate system before going on to some more difficult and interesting problems. For this case, it can be seen that the helical-coordinate treatment is no more difficult than the cylindrical-coordinate treatment, and this suggests that the former may be applied without loss, and perhaps with some gain, to problems previously treated in cylindrical coordinates.

4.3. *The helical wire*

We shall now consider the problem of wave propagation along a helical wire in otherwise free space. The wire is supposed to be infinitesimally thin, and we choose the helical coordinate system to have the same pitch as the wire. The equations of the wire can then be written

$$\left.\begin{aligned} r &= b \\ \zeta &= 0 \end{aligned}\right\}. \tag{34}$$

The boundary condition to be satisfied is that the tangential component of electric field on the boundary is zero, i.e.

$$E_\phi + E_\zeta \sin\alpha = 0. \tag{35}$$

Consider first E-waves. From equations (29), equation (35) becomes

$$(1-\beta^2/k^2)J_q(kb)e^{-j\beta 0}e^{-jn\phi}\sin\alpha = 0. \tag{36}$$

This condition can be satisfied in one of two ways, either by putting $J_q(kb) = 0$ or by adding a forward and a backward wave so as to replace $e^{-j\beta\zeta}$ with $\sin(2\pi m\zeta/p)$, m being an integer. This gives two sets of waves, and in general the field configuration will be given by adding members of both sets. If the conductor has extension in the surface $r = b$, the sine solutions vanish and we must choose $J_q(kb) = 0$. If the conductor has extension in the surface $\zeta = 0$, the condition $J_q(kb) = 0$ does not apply, and we must choose the sine solutions.

If a second conductor is present, we have a transmission line; an example of this is the slow-wave tube to be considered in section 4.4.

The case of H-waves is similar, except that the condition $J_q(kb) = 0$ that we have for E-waves is replaced by $J'_q(kb) = 0$.

4.4. *The slow-wave tube*

The idea of the slow-wave tube was first formulated by Kompfner (**2**, **3**). Many devices have since been invented for producing slow waves, but in Kompfner's form the slow-wave tube consists essentially of a helical wire of radius b and pitch p, concentric with a tube of radius a, a being greater than b. Electromagnetic waves may be thought of as travelling along the wire with phase velocity v_0, so that if the pitch angle of the wire is α_0 the component of velocity in the axial direction is

$$v_\phi = v_0 \sin\alpha_0. \tag{37}$$

Kompfner found experimentally that v_0 is approximately the velocity of light (**3**).

Since we have two distinct conductors, the device may be treated as a transmission line; it is possible, of course, to treat it as a waveguide, but as a slow-wave tube it functions as a line, and we shall only consider this case here. We shall proceed by analogy with the treatment of the coaxial line given by Lamont (**4**), and for simplicity assume that the wire is infinitesimally thin.

Anticipating Lamont, we assume a two-dimensional potential function ψ, of the form

$$\psi = \psi(r)e^{-jn\phi}. \tag{38}$$

For the electric field, we have

$$\mathbf{E} = -\operatorname{grad}\psi, \tag{39}$$

i.e.

$$\left.\begin{aligned} E_r &= -\frac{\partial\psi}{\partial r}e^{-jn\phi} \\ E_\phi &= \frac{jn\psi}{r\cos\alpha}e^{-jn\phi} \\ E_\zeta &= \frac{-jn\tan\alpha}{r}\psi e^{-jn\phi} \end{aligned}\right\}. \tag{40}$$

The magnetic field components could be obtained from Maxwell's equations by substitution from equation (40), but this is rather tedious and we do not require the results for our present purpose.

We require $\operatorname{div}\mathbf{E} = 0$. Substituting from equation (40) into equation (11), with $\mathbf{V}$ replaced by $\mathbf{E}$, and equating to zero, we obtain

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{n^2}{r^2}\psi = 0. \tag{41}$$

The solution of this is $\quad \psi = Ar^{\pm n}+\text{constant},$

and since ψ must remain finite at $r = 0$, we reject the negative sign. Also, the boundary conditions must be satisfied that E_ϕ and E_ζ vanish at $r = a$, which enables us to put $\psi = 0$ at $r = a$, i.e. the potential of the outer conductor is zero. Thus we write the solution of equation (41) in the form

$$\psi = a^n - r^n. \tag{42}$$

The function $\psi e^{-j\beta z}$ must also satisfy the wave equation for propagation in the z-direction. Now,

$$\psi e^{-j\beta z} = \psi(r)e^{-jn\phi}e^{-j\beta\zeta}e^{-jp\beta\phi/2\pi},$$

so that

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{(n+p\beta/2\pi)^2}{r^2}\psi-\beta^2\left[1+\frac{p^2}{4\pi^2r^2}\right]\psi+\frac{2p\beta}{2\pi r^2}\left[n+\frac{p\beta}{2\pi}\right]\psi+\frac{\omega^2}{v_z^2}\psi = 0, \tag{43}$$

where v_z is the velocity of electromagnetic waves in the z-direction. Subtracting equation (41) from equation (43), we obtain the characteristic equation

$$\beta^2 = \omega^2/v_z^2,$$

i.e.

$$v_z = \omega/\beta \tag{44}$$

as we should expect.

Here β is fixed by the requirement that at $\zeta = 0$ and $\zeta = p$ the fields take the same values, so that $p\beta/2\pi$ must be an integer, say ν, and we have

$$v_z = \omega p/2\pi\nu, \tag{45}$$

so that v_z is fixed by the frequency and the pitch of the helix. Equation (45) can only be expected to hold as long as p is less than $\nu\lambda_0$.

From equation (45), bearing in mind that $\omega\sqrt{(\epsilon_0\mu_0)} = \omega/c = 2\pi/\lambda_0$, where c is the velocity of light, we obtain

$$v_z = pc/\nu\lambda_0, \tag{46}$$

which shows that slow waves exist when $p < \nu\lambda_0$. The velocity v_z may be regarded as the z component of the velocity of waves travelling along the wire. We have

$$\sin\alpha_0 = \frac{p}{\sqrt{(4\pi^2b^2+p^2)}},$$

so that

$$v_0 = \frac{c\sqrt{(4\pi^2b^2+p^2)}}{\nu\lambda_0}. \tag{47}$$

Thus the waves travel along the wire with the velocity of light if the length of one turn of the helix is equal to $\nu\lambda_0$. Unfortunately, Kompfner's papers do not give enough information to enable us to check whether this condition was satisfied in his experiments.

4.5. *The helical waveguide*

This problem has been previously treated by the present author by an approximate method. The waveguide is bounded by the surfaces $r = b$, $r = a$, $\zeta = 0$, $\zeta = c$, c being less than p. We shall consider E-modes and H-modes separately.

E-modes

In equations (29) we put

$$X_q(kr) = J_q(kr)+AY_q(kr). \tag{48}$$

On $\zeta = 0$ and $\zeta = c$, $E_\phi+E_\zeta\sin\alpha = 0$. Thus we must take a forward wave and a backward wave and combine them so that in equations (29) $e^{-j\beta\zeta}$ is replaced by $\sin\nu\pi\zeta/c$ or $\cos\nu\pi\zeta/c$ as the case may be. We also have

$$k^2 = \omega^2\epsilon_0\mu_0-\beta^2 = \omega^2\epsilon_0\mu_0-\nu^2\pi^2/c^2. \tag{49}$$

The surfaces $\zeta = 0$ and $\zeta = c$ are equipotential surfaces, so that $\partial/\partial\phi = 0$, i.e. $n = 0$. The field equations thus become

$$\left.\begin{aligned}
E_r &= \frac{-\nu\pi}{kc}\{J'_q(kr)+AY'_q(kr)\}\sin\frac{\nu\pi\zeta}{c}\\
E_\phi &= \frac{-j\nu\pi q}{k^2rc\cos\alpha}\{J_q(kr)+AY_q(kr)\}\sin\frac{\nu\pi\zeta}{c}\\
E_\zeta &= \left(1+\frac{\nu pq}{2k^2r^2c}\right)\{J_q(kr)+AY_q(kr)\}\cos\frac{\nu\pi\zeta}{c}\\
H_r &= \frac{-\omega\epsilon_0 q}{k^2r}\{J_q(kr)+AY_q(kr)\}\cos\frac{\nu\pi\zeta}{c}\\
H_\phi &= \frac{-j\omega\epsilon_0}{k\cos\alpha}\{J'_q(kr)+AY'_q(kr)\}\cos\frac{\nu\pi\zeta}{c}\\
H_\zeta &= \frac{j\omega\epsilon_0 p}{2\pi kr}\{J'_q(kr)+AY'_q(kr)\}\cos\frac{\nu\pi\zeta}{c}
\end{aligned}\right\}. \tag{50}$$

We must also satisfy the condition that $(E_\phi + E_\zeta \sin\alpha)$ vanishes on $r = a$ and $r = b$, i.e.

$$J_q(ka) + AY_q(ka) = 0,$$
$$J_q(kb) + AY_q(kb) = 0.$$

These two equations must both give the same value for A, so that

$$\begin{vmatrix} J_q(ka) & Y_q(ka) \\ J_q(kb) & Y_q(kb) \end{vmatrix} = 0, \tag{51}$$

which is the characteristic equation, to be solved for q, since k is fixed by equation (49).

H-modes

These may be treated in the same way as E-modes. The characteristic equation is found to be

$$\begin{vmatrix} J'_q(ka) & Y'_q(ka) \\ J'_q(kb) & Y'_q(kb) \end{vmatrix} = 0, \tag{52}$$

and the field equations are

$$\left.\begin{aligned}
E_r &= \frac{\omega\mu_0 q}{k^2 r}\{J_q(kr) + A'Y_q(kr)\}\sin\left(\frac{\nu\pi\zeta}{c}\right) \\
E_\phi &= \frac{j\omega\mu_0}{k\cos\alpha}\{J'_q(kr) + A'Y'_q(kr)\}\sin\left(\frac{\nu\pi\zeta}{c}\right) \\
E_\zeta &= \frac{-j\omega\mu_0 p}{2\pi kr}\{J'_q(kr) + A'Y'_q(kr)\}\sin\left(\frac{\nu\pi\zeta}{c}\right) \\
H_r &= \frac{-\nu\pi}{kc}\{J'_q(kr) + A'Y'_q(kr)\}\cos\left(\frac{\nu\pi\zeta}{c}\right) \\
H_\phi &= \frac{-j\nu\pi q}{k^2 rc\cos\alpha}\{J_q(kr) + A'Y_q(kr)\}\cos\left(\frac{\nu\pi\zeta}{c}\right) \\
H_\zeta &= \left(1 + \frac{\nu p q}{2k^2 r^2 c}\right)\{J_q(kr) + A'Y_q(kr)\}\sin\left(\frac{\nu\pi\zeta}{c}\right)
\end{aligned}\right\}. \tag{53}$$

Now, from equations (2) and (23), remembering that $n = 0$, we may write

$$e^{-j\beta\zeta} = e^{-j\beta z}e^{jp\beta\phi/2\pi} = e^{-j\beta z}e^{-jq\phi}.$$

Thus q is the number of waves per turn of the helix. The characteristic equations (51) and (52) are identical in form with those for the infinite circular guide, which is a guide with surfaces $r = b$, $r = a$, $z = 0$, $z = c$, with the restriction removed that θ is limited to the range 0 to 2π. Thus the number of waves per turn in the helical guide is the same as in the infinite circular guide. This conclusion was reached by the author in the previous approximate treatment (**5**), where the method was to treat the infinite circular guide exactly and then apply perturbation theory to find

the change in the azimuthal phase constant (i.e. q) on shearing into the helical form. The change was found to be exactly zero.

The solution of the characteristic equations has been discussed by the author in the previous treatment, where tables are given of kb as a function of q, with the ratio a/b as a parameter. In using these tables, it should be remembered that the notation is different from that in the present paper, and care should be taken to avoid confusion.

4.6. *Coaxial transmission line with helical dielectric support*

The geometry of this system is similar to that of the helical waveguide treated in section 4.5; the cylindrical walls $r = a$, $r = b$ are conducting surfaces, and the region $0 < \zeta < c$ contains only air. The region $c < \zeta < p$ $(p > c)$, however, instead of being filled with metal as in the helical guide, contains a dielectric, which acts as a support for the inner conductor. The system may be treated as a waveguide or as a transmission line; the transmission line problem has been treated approximately by Griemsmann (**6**). We shall now give exact treatments of both the waveguide and the transmission line problems; the waveguide problem is of interest for the comparison between its characteristic equation and that of the helical waveguide.

(*a*) *Waveguide treatment*

The field equations take different forms according as we are dealing with the air- or dielectric-filled regions. For the air-filled region, we replace β by β_0 and k by k_0 in equations (29) and (30), and have

$$k_0^2 = \omega^2\epsilon_0\mu_0 - \beta_0^2. \tag{54}$$

In the dielectric-filled region, we replace β by β_1, k by k_1, and ϵ_0 by $\epsilon\epsilon_0$, and have

$$k_1^2 = \omega^2\epsilon_0\mu_0\epsilon - \beta_1^2. \tag{55}$$

We shall not write out the field equations in full at this stage. If we change the ζ coordinate but keep ϕ fixed, the phase change on traversing a distance p is $c\beta_0 + (p-c)\beta_1$. If we make the same change in position by keeping ζ fixed and changing ϕ by 2π, the phase change is $2\pi n$. Thus we have

$$c\beta_0 + (p-c)\beta_1 = 2\pi(n-q), \tag{56}$$

where q is an integer. We do not necessarily have the same phase at $\zeta_0 + p$ as at ζ_0, so that n will probably not be zero. Equation (56) may be compared with equation (23).

At $\zeta = 0$ and $\zeta = c$, the values of $E_\phi + E_\zeta \sin\alpha$ and E_r must be continuous, and this can only be the case if $k_0 = k_1$. Thus from equations (54) and (55)

$$\beta_1^2 = \beta_0^2 + \omega^2\epsilon_0\mu_0(\epsilon - 1). \tag{57}$$

In future we shall write k to mean either k_0 or k_1. From equations (56) and (57) we obtain

$$\beta_0 = 2\pi\{(n-q)c \pm (p-c)\sqrt{[(n-q)^2+p(\epsilon-1)(2c-p)/\lambda_0^2]}\} \tag{58}$$

and so k is a function of $(n-q)$ which we might write as

$$k = k(n-q). \tag{59}$$

At $r = a$ and $r = b$, $E_\phi + E_\zeta \sin\alpha$ must be zero, so that for E-modes we must have

$$\begin{vmatrix} J_q(ka) & Y_q(ka) \\ J_q(kb) & Y_q(kb) \end{vmatrix} = 0, \tag{60}$$

while for H-modes

$$\begin{vmatrix} J'_q(ka) & Y'_q(ka) \\ J'_q(kb) & Y'_q(kb) \end{vmatrix} = 0. \tag{61}$$

Equations (60) and (61) may be compared with equations (51) and (52). However, whereas the latter are to be solved for q, equations (60) and (61) are to be solved for k, q being assigned according to the mode. From equation (59), $(n-q)$, and therefore n, can be obtained, and we can then calculate β_0 and β_1 from equations (56) and (58). Unattenuated propagation will only occur if β_0 and β_1 are real, which, by virtue of equations (58) and (59), will impose a restriction on the solutions of equations (60) and (61) which are permissible. However, except for $q = 0$, β_0 and β_1 have real parts even in the attenuation regions, so that they are not true stop bands, and some propagation can take place.

(*b*) *Transmission line treatment*

As in the case of the slow-wave tube, we may follow a treatment analogous to that of Lamont (4) for the coaxial line with homogeneous support. We assume a potential function of the form

$$\psi = \psi(r, \phi) \tag{62}$$

and write

$$E = -\operatorname{grad}\psi\, e^{-j\beta\zeta}e^{-jn\phi}, \tag{63}$$

i.e.

$$\left.\begin{aligned} E_r &= -\frac{\partial\psi}{\partial r}e^{-j\beta\zeta}e^{-jn\phi} \\ E_\phi &= \frac{-1}{r\cos\alpha}\frac{\partial\psi}{\partial\phi}e^{-j\beta\zeta}e^{-jn\phi} \\ E_\zeta &= \frac{\tan\alpha}{r}\frac{\partial\psi}{\partial\phi}e^{-j\beta\zeta}e^{-jn\phi} \end{aligned}\right\}. \tag{64}$$

ψ must satisfy Laplace's equation, so that

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} = 0. \tag{65}$$

The wave equation is, in air,

$$(\nabla^2+\omega^2\epsilon_0\mu_0)(\psi e^{-j\beta_0 z}) = 0,$$

i.e.
$$(\nabla^2+\omega^2\epsilon_0\mu_0)(\psi e^{-j\beta_0\zeta}e^{-jp\beta_0\phi/2\pi}) = 0. \tag{66}$$

If we write out equation (66) in full, and subtract equation (65), we obtain

$$\beta_0^2 = \omega^2\epsilon_0\mu_0. \tag{67}$$

Similarly, for the dielectric region,

$$\beta_1^2 = \omega^2\epsilon_0\mu_0\epsilon. \tag{68}$$

These equations for β_0 and β_1 satisfy equation (57), as we should expect. Also, we may expect that ψ is everywhere independent of ϕ, as it must be on the two conducting surfaces $r = a$ and $r = b$. Thus E_ϕ and E_ζ are zero, so that there is no difficulty in satisfying the boundary condition, which is that E_r is continuous at the air-dielectric interface. The solution of equation (65) becomes

$$\psi = A\log r+B \tag{69}$$

as in the case of the homogeneously supported line.

The form of equations (67) and (68) shows that the behaviour of this line, as a line, is analogous to that of the coaxial line supported at regular intervals by annular disks of dielectric material. It is well known that when a line is loaded periodically, and when the wavelength of waves on the line is comparable with the spacing of the loads, a resonance effect occurs, so that the frequency spectrum is divided up into a series of pass-bands and stop-bands. This applies to the line with the annular disks, and therefore, by analogy, to the line with helical support. The calculation of the positions of the bands is easier for the cylindrical case than for the helical case, and in accordance with the theorem to be proved in section 6, we expect the results to be identical. It is not our intention to perform this calculation here; our purpose is to illustrate the uses of the helical coordinate system, not to give an exhaustive discussion of each topic mentioned. The theory of stop-bands and pass-bands is fully discussed by Brillouin (**7**).

5. Further applications

The helical coordinate system is applicable to any problem involving helical or cylindrical symmetry, although in the cylindrical case it may be that no new results are obtained. We shall indicate here a few problems which may usefully be discussed in helical coordinates, without, however, attempting such discussion.

A fluid flowing in a cylindrical tube may follow a helical path; this may be due to asymmetry in the pipe, causing rotation of one hand to be

preferred to that of the other. Other systems are the flow of air behind an air-screw, or of water behind a ship's propeller. It may be helpful to deal with screw dislocations in crystals by means of helical coordinates.

This coordinate system may also be useful in dealing with the motion of an electron in a magnetic field. An electron which initially has components of velocity parallel to and perpendicular to a magnetic field follows a helical path; this effect is made use of in the helical focusing of charged particles.

Note on toroidal helical coordinates

Imagine a circle Γ in a plane Π, and a line L in Π which does not intersect the circle. Move the circle parallel to L with constant velocity, and at the same time rotate Π about L with constant angular velocity. Let the velocity of the circle parallel to L be such that while Π undergoes a complete revolution the circle moves a distance greater than its diameter. The resulting figure is a helical toroid, related to the familiar circular toroid in the same way that the helical wire of section 4.3 is related to a circular wire. The coordinates appropriate to this figure are (i) the radius r measured from the centre of the circle, in the plane of the circle, (ii) the angle θ measured round the circle, (iii) the angle ϕ measured along the helical locus of a point in the circle, as it performs the motions described above. It is evident that the coordinate ϕ in this system is identical with the coordinate ϕ in the polar helical system with which this paper is chiefly concerned.

The toroidal helical coordinate system is appropriate to such problems as the helical waveguide of circular cross-section, which is probably of only academic interest, or the helical wire of finite thickness, which is a better model for Kompfner's slow-wave tube (section 4.4) than the helical wire of infinitesimal thickness which we have taken in this paper.

It is outside the scope of this paper to give an exhaustive discussion of toroidal helical coordinates; but it might be suggested here that, by analogy with the theorem proved in section 6, it is probably true that, for mathematical purposes, a helical toroidal system can be replaced by a circular toroidal system and still lead to the same results. Without going into the matter thoroughly, it is impossible to prove or disprove this statement, but intuitively one feels that it ought to be true. If so, then the helical wire of finite thickness can be replaced by a series of equally-spaced circular wires along a common axis, with the condition removed that on changing θ by 2π one returns to one's starting-point. This represents a great simplification of the problem, but the problem of solving the wave equation in (circular) toroidal coordinates still remains.

6. Discussion

In section 3.2 we introduced the idea of simply and multiply connected spaces. We can pursue the idea a little further now that we have seen the results of certain problems.

In the cylindrical waveguide (section 4.2), a change of p in ζ or of 2π in ϕ brings us to a point at which the fields are identical with those at the starting-point. The two changes are equivalent, so this is a case of multiply connected space.

In the helical waveguide, the surfaces $\zeta = 0$ and $\zeta =$ [illegible] constitute barriers; the changes are not equivalent. A change in ζ of p takes us outside the waveguide; a change in ϕ of 2π leaves us further along the guide, but at a point where, in general, the fields are not identical with those at our starting-point. This is an example of simply connected space.

The slow-wave tube and the coaxial line with helical support are rather different cases. The effect of changing ζ by p or ϕ by 2π is to bring us to the same point, but it is not an equivalent point, i.e. the fields are not the same as at the starting-point. These are examples of multiply connected space, and serve to demonstrate that in multiply connected space, the geometry of electromagnetic fields need not conform to the geometry of the material system.

From a consideration of the above cases, the following rules emerge:

(i) If it is possible to find a line parallel to the z-axis which does not enter a conducting medium, the system is multiply connected. This does not necessarily mean that all lines parallel to the z-axis must satisfy the condition; it is sufficient that it holds for at least some.

(ii) If it is not possible to find a line parallel to the z-axis which does not enter a conducting medium, the space is simply connected.

(iii) All helical spaces that occur in electromagnetic problems are either simply connected or multiply connected.

(iv) In multiply connected space, the periodicity of electromagnetic fields is not necessarily simply related to the pitch of the material system unless the system has cylindrical symmetry, in which case the pitch of the coordinate system can be chosen to give a simple relationship.

The practical usefulness of deciding what kind of space we are dealing with lies in the fact that for multiply connected space q is constrained to be an integer (including zero), and the characteristic equation has to be solved, effectively, for the wavelength parallel to ζ, i.e. for β. In simply connected space, on the other hand, β is constrained to be π times an integer, and the characteristic equation has to be solved, effectively, for the angular wavelength along the θ dimension, i.e. for q. A considerable

simplification of the treatment results if it is realized at the outset which of these cases applies to the problem under consideration.

We have seen that the helical waveguide has the same characteristic equations as the infinite circular guide, and that the treatment of the cylindrical waveguide in helical coordinates leads to the same characteristic equations as the treatment in cylindrical coordinates. Similarly, the helical wire in the slow-wave tube may be replaced by a series of circular wires, of radius b, concentric with the tube, and with equal spacing p from each other. The same characteristic equation will then result as in the case treated above (section 4.4). In Griemsmann's problem (section 4.6), the helical dielectric may be replaced by a series of annular disks, with the same thickness, $p-c$, as the actual dielectric, and regular spacing p. Again the same characteristic equation will result. In the last three cases it must be noted that in the cylindrical analogue, the condition that all fields repeat at intervals of 2π in θ, which usually applies to cylindrically symmetrical systems, does not apply; the cylindrical analogues are mathematical devices that cannot exist physically.

The remarks of the preceding paragraph suggest a result which we shall now prove.

THEOREM: *If a cylindrical analogue of a helical system be made by replacing the helical surfaces ζ = constant with planes z = constant$+sp$ (s an integer), and if in the case of simply connected space the requirement is not imposed that the conditions shall repeat at intervals of 2π in the azimuthal dimension θ, the same characteristic equation applies to the cylindrical system as to the helical system.*

To prove this, we write down first the field equations for the cylindrical system. They are:

$$\left.\begin{aligned} H_r &= \frac{1}{k^2}\left\{\frac{\partial}{\partial z}\left(\frac{\partial H_z}{\partial r}\right)+\frac{j\omega\epsilon_0}{r}\frac{\partial E_z}{\partial \theta}\right\} \\ H_\theta &= \frac{1}{k^2}\left\{\frac{1}{r}\frac{\partial}{\partial z}\left(\frac{\partial H_z}{\partial \theta}\right)-j\omega\epsilon_0\frac{\partial E_z}{\partial r}\right\} \\ E_r &= \frac{1}{k^2}\left\{\frac{-j\omega\mu_0}{r}\frac{\partial H_z}{\partial \theta}+\frac{\partial}{\partial z}\left(\frac{\partial E_z}{\partial r}\right)\right\} \\ E_\theta &= \frac{1}{k^2}\left\{j\omega\mu_0\frac{\partial H_z}{\partial r}+\frac{1}{r}\frac{\partial}{\partial z}\left(\frac{\partial E_z}{\partial \theta}\right)\right\} \end{aligned}\right\}. \tag{70}$$

The corresponding expressions for the helical system are given by equations (28), and if we bear in mind equations (9), where $\mathbf{V}$ may be $\mathbf{E}$ or $\mathbf{H}$, it becomes apparent that any condition on E_r, $E_\phi+E_\zeta\sin\alpha$, H_r, or $H_\phi+H_\zeta\sin\alpha$ in the helical system must lead to the corresponding restric-

tion on E_r, E_θ, H_r, or H_θ, respectively, in the analogous cylindrical system. Boundary conditions involving E_ζ or H_ζ always involve them in combination with E_ϕ or H_ϕ as $E_\zeta + E_\phi \sin\alpha$ or $H_\zeta + H_\phi \sin\alpha$, which are identical with E_z and H_z. Comparing equations (28) and (70), it is apparent that boundary conditions in the helical system must lead to the same equations in E_z and H_z as the corresponding boundary conditions in the cylindrical system. The theorem is thus proved for electromagnetic waves; an analogous proof could be given for any other kind of wave motion.

The application of this theorem leads to a great simplification of the problem of deriving the characteristic equation for a helical system, but the field components and normal modes are better discussed in helical coordinates. However, the most difficult problem is always to derive and solve the characteristic equation, so that anything which simplifies this process is valuable.

To sum up this discussion, the best way to treat a helical system is to decide first what kind of space is appropriate, and describe the cylindrical analogue. The characteristic equation for the analogue may then be obtained, and this is the same as the characteristic equation for the actual helical system. Having solved the characteristic equation, the field equations for the actual system can be obtained by substitution in equations (29) and (30), or, more generally, (28).

Acknowledgement

The author wishes to express his gratitude to Marconi's Wireless Telegraph Co., Ltd., for permission to publish this paper.

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TORSION AND FLEXURE OF SOLID CYLINDERS WITH CROSS-SECTIONS TRANSFORMABLE TO A RING-SPACE

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[Received 29 November 1956.—Revise received 14 November 1957]

SUMMARY

This paper deals with the torsion and flexure of a solid cylinder of simply-connected boundary cross-section which can be mapped on the ring-space between two concentric circles, the inner one being the transform of an inner slit or cut in the cross-section which, however, is not a physical boundary of the cylinder. The solutions for the potential functions, torsional constant, and centre of flexure are determined in the general case and particular values of these are found for the elliptic cylinder and a cylinder having a circular arc as internal cut.

1. Introduction

IN his book, *Some Basic Problems of the Mathematical Theory of Elasticity*, Muskhelishvili (**1**) deals, *inter alia*, with the fundamental boundary value problems of two-dimensional elasticity using various methods, and the case of the fundamental problem for a continuous ellipse is dealt with by means of conformal mapping on to a ring-space and by using infinite series solutions. The Cauchy integral methods described by Muskhelishvili and used also by Sokolnikoff (**2**) for the torsion and flexure problem cannot be applied to the particular case of the ellipse when one uses the usual conformal transformation of the form

$$z = z(t) = K\left(t+\frac{m}{t}\right) \quad (K > 0,\, m > 0),$$

transforming the space inside the ellipse on to the ring-space between two circles, since $z'(t) = 0$ at points inside the circle corresponding to the ellipse. The results for the torsion and flexure problem for the ellipse, as examples of the fundamental boundary value problem have not been given by Muskhelishvili, and Sokolnikoff determines them by means of real function theory methods which have been known for some considerable time. Complex variable methods using conformal transformations have been used by Stevenson (**3**) and Sherman (**4**) for problems of plane stress involving a thin isotropic elliptic plate, and one of the present

authors (Morris) has also developed complex variable methods for two-dimensional potential theory problems. The results of Morris (5) for Saint-Venant's torsion and flexure problem were published in three papers subsequently referred to as (I), (II), (III), and the paper (II) dealt specifically with cylinders of elliptic type for which the transformations used had an internal cut joining branch points of the transformation.

It was in dealing with aeolotropic plate theory problems later, that Morris and Livens (6, 7) had to consider again the question of choosing potential functions which were non-cyclic in a region containing singular points of the transformations used for the particular shaped plates. The problem was similar to the torsion and flexure problem as dealt with in (II) and at first it was thought that the method of that paper could be used. However, it was discovered that this method did not yield the correct result (verified by reduction to the isotropic case), and a new method had to be evolved. This method for the plane stress problem of a thin elliptic aeolotropic plate under given edge conditions is given in the paper by Morris (7). Morris realized immediately the possibility that the results of (II) might be incorrect, and on examining these it was soon found that this was the case. The object of this paper is to indicate that error and by application of the new method, correct the subsequent results of (II).

The new method will be shown to be applicable to any boundary section which is such that the double-Riemann surface inside the cross-section having a cut along a given curve inside the cross-section is transformable to a ring-space between two concentric circles, which we define in the t-plane by $|t| = 1$, $|t| = e^{-2\alpha}$. The cut inside the cross-section corresponds to the circle $|t| = e^{-\alpha}$ so that with $|t| = 1$ on the boundary the cross-section inside the boundary in the z-plane corresponds to the ring-space $e^{-\alpha} \leqslant |t| \leqslant 1$ in the t-plane. The particular cases of the elliptic cross-section and a cross-section having a circular arc as internal cut are treated. The notation of (II) is followed unless otherwise stated with $t \equiv e^{-i\zeta}$.

2. The new method

It was stated in (II), section 4, that the regularity of the potential functions could be secured provided the gradients of the potentials do not become infinite at the singular points of the transformation. This condition was secured and a solution for the complex potential function Ω_1 determined as

$$\Omega_1 = \sum_{r=1}^{\infty} \{\Omega_{r,1} e^{ri\zeta} + \Omega_{-r,1} e^{-ri\zeta}\} + \tfrac{1}{2}(1+2\sigma)z^2,$$

where
$$\Omega_{r,1} = \tfrac{1}{4}(1+2\sigma)\bar{f}_{-r}+\tfrac{1}{2}(g_r-\bar{g}_{-r})-\sigma\bar{g}_{-r},$$
$$\Omega_{-r,1} = \tfrac{1}{4}(1+2\sigma)\bar{f}_r+\tfrac{1}{2}(g_{-r}-\bar{g}_r)-\sigma\bar{g}_r,$$
the coefficients f_r, g_r, etc., depend on the shape of the cross-section, and σ is Poisson's ratio. The relationship between this function Ω_1 and the potential function χ given by Love (**8**) is
$$\chi = \mathrm{re}[\tfrac{1}{6}(2+\sigma)z^3-\Omega_1],$$
and when the appropriate values of f_r, g_r, etc., for the elliptic cross-section are substituted in the above value of Ω_1, this is found not to agree with the value of χ given by Love for the ellipse. This indicates the error in (II) and the consequent incorrectness of the results in (II).

The required condition is that the value of the potential must remain unchanged in crossing the cut joining the branch points of the transformation. From the papers by Livens and Morris (**6**), and Morris (**7**) it is noted that if a characteristic boundary condition for the potential function $\Omega(t)$ is of the form $\sum_{-\infty}^{\infty} B_n t_0^n$ (where $t = t_0 = e^{-i\xi}$ on the boundary), then a suitable form for $\Omega(t)$ is $\sum_{-\infty}^{\infty} A_n t^n$, where the A_n's are constants to be determined from the boundary conditions. But for any function, $z = z(t)$, of t it is known that if there are two branch points in the region with which we are concerned, then in going round one of the branch points (across the cut joining the points) the two roots of t, as a function of z, are interchanged. Thus, in particular, if t is the one root which reduces to t_0 on the boundary, and the other root is $e^{-2\alpha}/t$ then by writing
$$\Omega(t) = A_0+\sum_{n=1}^{\infty} A_n e^{n\alpha}(t^n+e^{-2n\alpha}/t^n),$$
i.e.
$$\Omega(t) = A_0+\sum_{n=1}^{\infty} A_n(e^{n\alpha}t^n+e^{-n\alpha}t^{-n}), \tag{2.1}$$
it may be ensured that in such a passage round one of the branch points the value of $\Omega(t)$ remains unchanged.

In effect this is equivalent to saying that in crossing the slit in the z-plane from P to P' say, since these adjacent points P and P' transform to conjugate points t, $\bar{t}$ on the circle $|t| = e^{-\alpha}$, or $t\bar{t} = e^{-2\alpha}$, corresponding to the slit, then the potential function must have the same value when t is changed to $\bar{t}$ where $t\bar{t} = e^{-2\alpha}$.

Thus if
$$\Omega(t) = \sum_{-\infty}^{\infty} A_n e^{n\alpha}t^n,$$
we must have
$$\sum_{-\infty}^{\infty} A_n e^{n\alpha}t^n = \sum_{-\infty}^{\infty} A_n e^{n\alpha}\bar{t}^n = \sum_{-\infty}^{\infty} A_n e^{-n\alpha}t^{-n} = \sum_{-\infty}^{\infty} A_{-n} e^{n\alpha}t^n,$$

so that $A_{-n} = A_n$ and the form (2.1) is correct. In considering this, however, in terms of the different roots of $z = z(t)$ it is possible to see how the method can be extended to the more complicated transformations when there are more than two branch points inside the boundary, as, for example, in cases (iii), (iv), (v), (vi), of (II), section 3.

This method for determining single-valued potential functions must now be applied to the potential functions required for the flexure and torsion problem in the particular cases chosen. It is first necessary to obtain the boundary values of these functions in terms of $t_0 = e^{-i\xi}$ rather than in terms of ζ as given in (II).

The first complete potential Ω_1 is defined (II, p. 7) by

$$\Omega_1 = (1+\sigma)\Omega_{11} - \sigma\Omega_{21} = (1+\sigma)\{\phi_{11}+i\psi_{11}\} - \sigma\{\phi_{21}+i\psi_{21}\},$$

and in terms of ξ and η the boundary conditions (I, pp. 86–87) are

$$\frac{\partial}{\partial\eta}\left[\phi_{11} - \tfrac{1}{24}\{z(\zeta)+\bar{z}(\bar{\zeta})\}^3\right]_{\eta=0} = 0 \tag{2.2}$$

and
$$\psi_{21} = \frac{i}{24}\{z(\xi)-\bar{z}(\xi)\}^3 + \text{const.} \tag{2.3}$$

(2.2) becomes

$$\left[\frac{\partial\phi_{11}}{\partial\eta}\right]_{\eta=0} = \left[\tfrac{1}{8}\{z(t)+\bar{z}(\bar{t})\}^2\left\{z'(t)\frac{\partial t}{\partial\eta} + \bar{z}'(\bar{t})\frac{\partial\bar{t}}{\partial\eta}\right\}\right]_{\eta=0}.$$

Now
$$\left[\frac{\partial\phi_{11}}{\partial\eta}\right]_{\eta=0} = \left[-\frac{\partial\psi_{11}}{\partial\xi}\right]_{\eta=0} = \left[-\frac{\partial\psi_{11}}{\partial t}\frac{\partial t}{\partial\xi}\right]_{\eta=0} = it_0\left[\frac{\partial\psi_{11}}{\partial t}\right]_{t=t_0}$$

and thus

$$it_0\left[\frac{\partial\psi_{11}}{\partial t}\right]_{t=t_0} = \left[\tfrac{1}{8}\{z(t)+\bar{z}(\bar{t})\}^2\left\{\frac{d}{dt}z(t)\frac{\partial t}{\partial\eta} + \frac{d}{d(1/t)}\bar{z}\left(\frac{1}{t}\right)\frac{\partial\bar{t}}{\partial\eta}\right\}\right]_{\eta=0}$$

or
$$it_0\left[\frac{\partial\psi_{11}}{\partial t}\right]_{t=t_0} = \left[\frac{1}{8}\left\{z(t)+\bar{z}\left(\frac{1}{t}\right)\right\}^2\left\{t\frac{d}{dt}z(t) - t\frac{d}{dt}\bar{z}\left(\frac{1}{t}\right)\right\}\right]_{t=t_0}$$

so that on the boundary

$$\psi_{11} = \frac{1}{8i}\int^{t_0}\left\{z(t)+\bar{z}\left(\frac{1}{t}\right)\right\}^2\left\{z'(t)-\bar{z}'\left(\frac{1}{t}\right)\right\}dt, \tag{2.4}$$

where in (2.4) dashes denote differentiation with respect to t.

Equation (2.3) may be written in the form

$$\psi_{21} = \frac{i}{24}\left[z(t)-\bar{z}\left(\frac{1}{t}\right)\right]^3_{t=t_0} + \text{const},$$

and from this and (2.4) and the fact that ψ_{11} and ψ_{21} are respectively the

imaginary parts of Ω_{11} and Ω_{21}, we may form our boundary condition, viz.:

$$\Omega_1(t_0)-\bar{\Omega}_1\left(\frac{1}{t_0}\right) = (1+\sigma)\left\{\Omega_{11}(t_0)-\bar{\Omega}_{11}\left(\frac{1}{t_0}\right)\right\}-\sigma\left\{\Omega_{21}(t_0)-\bar{\Omega}_{21}\left(\frac{1}{t_0}\right)\right\}$$

$$= 2i[(1+\sigma)\psi_{11}-\sigma\psi_{21}]_{t=t_0}$$

$$= \frac{(1+\sigma)}{4}\int^{t_0}\left\{z(t)+\bar{z}\left(\frac{1}{t}\right)\right\}^2\left\{z'(t)-\bar{z}'\left(\frac{1}{t}\right)\right\}\,dt+\frac{\sigma}{12}\left\{z(t_0)-\bar{z}\left(\frac{1}{t_0}\right)\right\}^3$$

which may be rewritten as

$$\Omega_1(t_0)-\bar{\Omega}_1\left(\frac{1}{t_0}\right) = \frac{1+2\sigma}{12}\left\{z(t_0)-\bar{z}\left(\frac{1}{t_0}\right)\right\}^3+(1+\sigma)\int^{t_0} z(t)\bar{z}\left(\frac{1}{t}\right)\left\{z'(t)-\bar{z}'\left(\frac{1}{t}\right)\right\}\,dt, \tag{2.5}$$

where dashes indicate differentiation with respect to t.

Similarly the boundary conditions on Ω_2, Ω_3 are

$$\Omega_2(t_0)-\bar{\Omega}_2\left(\frac{1}{t_0}\right) = \frac{i(1+2\sigma)}{12}\left\{z(t_0)+\bar{z}\left(\frac{1}{t_0}\right)\right\}^3-i(1+\sigma)\int^{t_0} z(t)\bar{z}\left(\frac{1}{t}\right)\left\{z'(t)+\bar{z}'\left(\frac{1}{t}\right)\right\}\,dt, \tag{2.6}$$

and

$$\Omega_3(t_0)-\bar{\Omega}_3\left(\frac{1}{t_0}\right) = iz(t_0)\bar{z}\left(\frac{1}{t_0}\right), \tag{2.7}$$

respectively, constants having been omitted.

3. Application of the method to the particular class of cross-section

Let
$$z = z(t) = \sum_{n=-\infty}^{\infty} a_n t^n, \qquad t = e^{-i\zeta}, \tag{3.1}$$

be the transformation which transforms the double Riemann surface having a given cut inside the solid section of the cylinder into the ring-space between the two concentric circles $|t| = 1$ and $|t| = e^{-2\alpha}$, so that the two roots of $z = z(t)$ are t, which reduces to $t_0 = e^{-i\xi}$ on the boundary, and $e^{-2\alpha}/t$ as stated in § 2.

The constant a_0 in (3.1) is first adjusted so that the origin is at the centroid of the cross-section. The position z_G of the centroid is ascertained, as in (II), from

$$\left.\begin{aligned} S &= -\tfrac{1}{2}i\int_C \bar{z}\,dz = -\tfrac{1}{2}i\int_{C_1}\bar{z}\left(\frac{1}{t}\right)z'(t)\,dt \\ z_G S &= -\tfrac{1}{2}i\int_C z\bar{z}\,dz = -\tfrac{1}{2}i\int_{C_1} z(t)\bar{z}\left(\frac{1}{t}\right)z'(t)\,dt \end{aligned}\right\}, \tag{3.2}$$

where the integrals in the t-plane are taken around the unit circle C_1.

Bearing in mind the conditions (2.5, 6, 7) we assume that our boundary condition takes the form

$$\Omega(t_0)-\bar{\Omega}\left(\frac{1}{t_0}\right) = \sum_{n=-\infty}^{\infty} B_n t_0^n, \tag{3.3}$$

where we note that $\bar{B}_n = -B_n$. Then writing $\Omega(t)$ in the form (2.1), viz.

$$\Omega(t) = A_0 + \sum_{n=1}^{\infty} A_n(e^{n\alpha}t^n + e^{-n\alpha}t^{-n}),$$

and using (3.3) we find that the arbitrary constants A_n must be such that

$$A_0-\bar{A}_0 = B_0,$$

$$A_n e^{n\alpha}-\bar{A}_n e^{-n\alpha} = B_n,$$

$$A_n e^{-n\alpha}-\bar{A}_n e^{n\alpha} = B_{-n},$$

and thus

$$\left.\begin{aligned} A_0-\bar{A}_0 = B_0, \quad \text{and for } n > 0 \\ A_n = \tfrac{1}{2}(e^{n\alpha}B_n - e^{-n\alpha}B_{-n})\operatorname{cosech} 2n\alpha \end{aligned}\right\}. \tag{3.4}$$

To determine the coefficients B_n in each of the boundary conditions we make use of the following notation:

$$z(t)\bar{z}\left(\frac{1}{t}\right) = \sum_{n=-\infty}^{\infty} b_n t^n, \tag{3.5}$$

$$z(t)\bar{z}\left(\frac{1}{t}\right)z'(t) = \sum_{n=-\infty}^{\infty} c_n t^{n-1}, \tag{3.6}$$

where

$$b_n = \sum_{r=-\infty}^{\infty} a_{n+r}\bar{a}_r, \quad \text{with} \quad b_{-n} = \bar{b}_n \tag{3.7}$$

and

$$c_n = \sum_{r=-\infty}^{\infty} (n+r)a_{n+r}b_{-r}, \tag{3.8}$$

which with the origin at the centroid of the cross-section implies that $c_0 = 0$. It then follows that

$$z(t)\bar{z}\left(\frac{1}{t}\right)\bar{z}'\left(\frac{1}{t}\right) = -\frac{1}{t^2}\sum_{n=-\infty}^{\infty} \bar{c}_n\left(\frac{1}{t}\right)^{n-1} = -\sum_{n=-\infty}^{\infty} \bar{c}_n t^{-(n+1)}. \tag{3.9}$$

We also use

$$\left\{z(t)-\bar{z}\left(\frac{1}{t}\right)\right\}^3 = \sum_{n=-\infty}^{\infty} g_n t^n, \tag{3.10}$$

and

$$\left\{z(t)+\bar{z}\left(\frac{1}{t}\right)\right\}^3 = \sum_{n=-\infty}^{\infty} h_n t^n, \tag{3.11}$$

where

$$g_n = \sum_{r=-\infty}^{\infty} f_{n+r}(a_{-r}-\bar{a}_r), \tag{3.12}$$

and

$$f_n = \sum_{r=-\infty}^{\infty} (a_{n+r}-\bar{a}_{-(n+r)})(a_{-r}-\bar{a}_r) \tag{3.13}$$

with $a_0 = a_{-0}$. Also $$h_n = \sum_{r=-\infty}^{\infty} j_{n+r}(a_{-r}+\bar{a}_r) \tag{3.14}$$

and $$j_n = \sum_{r=-\infty}^{\infty} (a_{n+r}+\bar{a}_{-(n+r)})(a_{-r}+\bar{a}_r). \tag{3.15}$$

4. The three complex potential functions

The boundary condition for the first complex potential function is, from (2.5),

$$\Omega_1(t_0)-\bar{\Omega}_1\left(\frac{1}{t_0}\right) = \frac{(1+2\sigma)}{12}\left\{z(t_0)-\bar{z}\left(\frac{1}{t_0}\right)\right\}^3+(1+\sigma)\int^{t_0} z(t)\bar{z}\left(\frac{1}{t}\right)\left\{z'(t)-\bar{z}'\left(\frac{1}{t}\right)\right\}dt,$$

which with the above notation reduces to

$$\Omega_1(t_0)-\bar{\Omega}_1\left(\frac{1}{t_0}\right) = \sum_{n=-\infty}^{\infty} B_{1,n}\, t_0^n,$$

where $$B_{1,n} = \frac{(1+2\sigma)}{12}g_n+\frac{(1+\sigma)}{n}(c_n+\bar{c}_{-n}).$$

Thus $$\Omega_1(t) = A_{1,0}+\sum_{n=1}^{\infty} A_{1,n}(e^{n\alpha}t^n+e^{-n\alpha}t^{-n}) \tag{4.1}$$

and we have from equations (3.4)

$$\left.\begin{aligned} &A_{1,0}-\bar{A}_{1,0} = \tfrac{1}{12}(1+2\sigma)g_0, \quad \text{and for } n>0 \\ &\quad A_{1,n} = \tfrac{1}{2}(\operatorname{cosech} 2n\alpha)\left[\frac{(1+2\sigma)}{12}(e^{n\alpha}g_n-e^{-n\alpha}g_{-n})+\right. \\ &\qquad\qquad \left.+\frac{(1+\sigma)}{n}\{e^{n\alpha}(c_n+\bar{c}_{-n})+e^{-n\alpha}(c_{-n}+\bar{c}_n)\}\right] \end{aligned}\right\}. \tag{4.2}$$

Similarly, from (2.6),

$$\Omega_2(t_0)-\bar{\Omega}_2\left(\frac{1}{t_0}\right) = \sum_{n=-\infty}^{\infty} B_{2,n}\, t_0^n,$$

where $$B_{2,n} = \frac{i(1+2\sigma)}{12}h_n-\frac{i(1+\sigma)}{n}(c_n-\bar{c}_{-n}),$$

so that again if

$$\Omega_2(t) = A_{2,0}+\sum_{n=1}^{\infty} A_{2,n}(e^{n\alpha}t^n+e^{-n\alpha}t^{-n}), \tag{4.3}$$

we have

$$\left.\begin{aligned} &A_{2,0}-\bar{A}_{2,0} = \frac{i(1+2\sigma)}{12}h_0, \quad \text{and for } n>0 \\ &\quad A_{2,n} = \tfrac{1}{2}(\operatorname{cosech} 2n\alpha)\left[\frac{i(1+2\sigma)}{12}(e^{n\alpha}h_n-e^{-n\alpha}h_{-n})-\right. \\ &\qquad\qquad \left.-\frac{i(1+\sigma)}{n}\{e^{n\alpha}(c_n-\bar{c}_{-n})+e^{-n\alpha}(c_{-n}-\bar{c}_n)\}\right] \end{aligned}\right\}. \tag{4.4}$$

Similarly from (2.7)

$$\Omega_3(t_0)-\bar{\Omega}_3\left(\frac{1}{t_0}\right) = \sum_{n=-\infty}^{\infty} ib_n\, t_0^n,$$

so that if
$$\Omega_3(t) = A_{3,0}+\sum_{n=1}^{\infty} A_{3,n}(e^{n\alpha}t^n+e^{-n\alpha}t^{-n}), \tag{4.5}$$
then
$$\left.\begin{aligned} A_{3,0}-\bar{A}_{3,0} &= ib_0, \quad \text{and for} \quad n>0 \\ A_{3,n} &= \tfrac{1}{2}i\{e^{n\alpha}b_n-e^{-n\alpha}b_{-n}\}\operatorname{cosech} 2n\alpha \end{aligned}\right\}. \tag{4.6}$$

5. The torsional constant and the centre of flexure

Following Morris (I, p. 93), we have that the torsional constant Γ is the real part of
$$\Gamma+i\Gamma^* = \tfrac{1}{4}\tau\mu\int\limits_C \left(2\frac{d\Omega_3}{dz}-i\bar{z}\right)z\bar{z}\,dz,$$
but, writing in the notation of (I)
$$\bar{Z}_z = Z_x-iZ_y = \mu\tau\left(\frac{d\Omega_3}{dz}-i\bar{z}\right),$$
we see that
$$\Gamma+i\Gamma^* = \tfrac{1}{4}\int\limits_C z\bar{z}(2\bar{Z}_z+i\bar{z})\,dz,$$
so that
$$2i\Gamma^* = \tfrac{1}{2}\int\limits_C z\bar{z}(\bar{Z}_z\,dz-Z_z\,d\bar{z})+\tfrac{1}{4}i\int\limits_C z\bar{z}\,d(z\bar{z}),$$
and this is zero since on the boundary C the condition
$$Z_n = Z_x\frac{dy}{ds}-Z_y\frac{dx}{ds} = 0$$
makes
$$\bar{Z}_z dz-Z_z d\bar{z} = 0.$$
Thus the integral is real and
$$\Gamma = \tfrac{1}{4}\tau\mu\int\limits_C \left(2\frac{d\Omega_3}{dz}-i\bar{z}\right)z\bar{z}\,dz.$$
This integral being interpreted in the form
$$\tfrac{1}{4}\tau\mu\int\limits_{C_1}\left\{2\frac{d\Omega_3}{dt}-i\bar{z}\left(\frac{1}{t}\right)z'(t)\right\}z(t)\bar{z}\left(\frac{1}{t}\right)dt,$$
the general result for Γ is
$$\Gamma = \tfrac{1}{2}\pi\mu\tau i\left\{\sum_{n=1}^{\infty} 2nA_{3,n}(e^{n\alpha}b_{-n}-e^{-n\alpha}b_n)-i\sum_{n=-\infty}^{\infty} c_n\bar{a}_n\right\}. \tag{5.1}$$

Following Morris (II, p. 12), but with a slightly different notation, the position of the centre of flexure is given by (x_0, y_0) where x_0 is the real part of
$$-\pi\mu iE^{-1}(AB-H^2)^{-1}\sum_{n=1}^{\infty}[nb_n\{A\Omega_{2,-n}-H\Omega_{1,-n}\}-nb_{-n}\{A\Omega_{2,n}-H\Omega_{1,n}\}]-$$
$$-\tfrac{1}{8}\pi\mu E^{-1}(AB-H^2)^{-1}(A-iH)(1-2\sigma)\sum_{n=-\infty}^{\infty} c_{-n}b_n,$$

and y_0 is the real part of

$$-\pi\mu iE^{-1}(AB-H^2)^{-1}\sum_{n=1}^{\infty}[nb_n(H\Omega_{2,-n}-B\Omega_{1,-n})+nb_{-n}(B\Omega_{1,n}-H\Omega_{2,n})]+$$
$$+\tfrac{1}{8}\pi\mu iE^{-1}(AB-H^2)^{-1}(B+iH)(1-2\sigma)\sum_{n=-\infty}^{\infty}b_n c_{-n},$$

where E is Young's modulus, and $\Omega_{1,n}$, $\Omega_{1,-n}$ are respectively the coefficients of t^n, t^{-n} in the expansion of $\{\Omega_1(t)-\frac{1}{12}(1+2\sigma)z^3(t)\}$, whilst $\Omega_{2,n}$, $\Omega_{2,-n}$ are respectively the coefficients of t^n, t^{-n} in the expansion of

$$\Omega_2(t)-\frac{i}{12}(1+2\sigma)z^3(t).$$

Also A, B, H are the usual moments and product of inertia of the cross-section and are given by

$$A+B=-\tfrac{1}{4}i\int_C z\bar{z}^2\,dz=-\tfrac{1}{4}i\int_{C_1}z(t)\bar{z}^2\left(\frac{1}{t}\right)z'(t)\,dt,$$

$$B-A-2iH=\tfrac{1}{2}i\int_C z^2\bar{z}\,dz=\tfrac{1}{2}i\int_{C_1}z^2(t)\bar{z}\left(\frac{1}{t}\right)z'(t)\,dt.$$

For principal axes, $H=0$, and we have that x_0 is the real part of

$$-\pi\mu i(EB)^{-1}\sum_{n=1}^{\infty}\{nb_n\Omega_{2,-n}-nb_{-n}\Omega_{2,n}\}-\tfrac{1}{8}\pi\mu(EB)^{-1}(1-2\sigma)\sum_{n=-\infty}^{\infty}b_n c_{-n}, \tag{5.2}$$

y_0 is the real part of

$$\pi\mu i(AE)^{-1}\sum_{n=1}^{\infty}\{nb_n\Omega_{1,-n}-nb_{-n}\Omega_{1,n}\}+\tfrac{1}{8}\pi\mu i(AE)^{-1}(1-2\sigma)\sum_{n=-\infty}^{\infty}b_n c_{-n}, \tag{5.3}$$

and

$$A=\tfrac{1}{4}\pi\sum_{n=-\infty}^{\infty}c_n\bar{a}_n+\tfrac{1}{2}\pi\sum_{n=-\infty}^{\infty}c_n a_{-n},$$

$$B=\tfrac{1}{4}\pi\sum_{n=-\infty}^{\infty}c_n\bar{a}_n-\tfrac{1}{2}\pi\sum_{n=-\infty}^{\infty}c_n a_{-n}.$$

To evaluate $\Omega_{1,n}$, $\Omega_{2,n}$ we use the notation

$$z^3(t)=\sum_{n=-\infty}^{\infty}l_n t^n, \tag{5.4}$$

where $$l_n=\sum_{r=-\infty}^{\infty}a_{n+r}d_{-r},$$

and $$d_n=\sum_{r=-\infty}^{\infty}a_{n+r}a_{-r}, \tag{5.5}$$

and then $$\Omega_{1,n}=A_{1,n}e^{n\alpha}-\tfrac{1}{12}(1+2\sigma)l_n, \tag{5.6}$$

$$\Omega_{2,n}=A_{2,n}e^{n\alpha}-\tfrac{1}{12}i(1+2\sigma)l_n. \tag{5.7}$$

6. The elliptic cross-section

In the special case of the ellipse we use the transformation

$$z = z(t) = \tfrac{1}{2}c\{e^{\alpha}t+e^{-\alpha}t^{-1}\}, \tag{6.1}$$

where the double Riemann surface within the ellipse in the z-plane is transformed to the ring-space between the two concentric circles in the t-plane.

Comparing (6.1) with (3.1), i.e. $z = \sum_{n=-\infty}^{\infty} a_n t^n$, we have immediately all the results for the ellipse if we substitute in the formulae of sections 3 and 5 the values

$$\left.\begin{array}{c} a_1 = \tfrac{1}{2}ce^{\alpha}, \qquad a_{-1} = \tfrac{1}{2}ce^{-\alpha} \\ a_n = 0 \quad (n \neq \pm 1) \end{array}\right\}. \tag{6.2}$$

The required coefficients b_n, c_n, etc., are

$$\left.\begin{array}{c} b_0 = \tfrac{1}{2}c^2\cosh 2\alpha, \qquad b_2 = b_{-2} = \tfrac{1}{4}c^2 \\ b_n = 0 \quad (n \neq 0, \pm 2) \end{array}\right\}. \tag{6.3}$$

$$\left.\begin{array}{c} c_1 = \tfrac{1}{8}c^3e^{3\alpha}, \qquad c_{-1} = -\tfrac{1}{8}c^3e^{-3\alpha} \\ c_3 = \tfrac{1}{8}c^3e^{\alpha}, \qquad c_{-3} = -\tfrac{1}{8}c^3e^{-\alpha} \\ c_n = 0 \quad (n \neq \pm 1, \pm 3) \end{array}\right\}. \tag{6.4}$$

$$\left.\begin{array}{c} g_1 = -g_{-1} = -\tfrac{3}{8}c^3(e^{\alpha}-e^{-\alpha})^3 \\ g_3 = -g_{-3} = \tfrac{1}{8}c^3(e^{\alpha}-e^{-\alpha})^3 \\ g_n = 0 \quad (n \neq \pm 1, \pm 3) \end{array}\right\}. \tag{6.5}$$

$$\left.\begin{array}{c} h_1 = h_{-1} = \tfrac{3}{8}c^3(e^{\alpha}+e^{-\alpha})^3 \\ h_3 = h_{-3} = \tfrac{1}{8}c^3(e^{\alpha}+e^{-\alpha})^3 \\ h_n = 0 \quad (n \neq \pm 1, \pm 3) \end{array}\right\}. \tag{6.6}$$

Since by symmetry the centre of flexure is at the centre of the cross-section it is not necessary to evaluate any of the constants for the determination of (x_0, y_0).

Using these results we determine the constants $A_{m,n}$ $(m = 1,2,3)$ in the potential functions as follows. From (4.2)

$$A_{1,0}-\bar{A}_{1,0} = 0,$$

$$A_{1,1} = \tfrac{1}{8}c^3[(1+\sigma)(2\cosh 2\alpha+1)-(1+2\sigma)\sinh^2\alpha],$$

$$A_{1,3} = \frac{c^3}{24(2\cosh 2\alpha+1)}[(1+\sigma)+(1+2\sigma)\sinh^2\alpha],$$

$$A_{1,n} = 0 \quad (n \neq 0, 1, 3).$$

Thus

$$\Omega_1(t) = A_{1,0}+A_{1,1}(e^{\alpha}t+e^{-\alpha}t^{-1})+A_{1,3}(e^{3\alpha}t^3+e^{-3\alpha}t^{-3}),$$

$$= A_{1,0}+A_{1,1}\frac{2}{c}z(t)+A_{1,3}\frac{8}{c^3}\left\{z^3(t)-\frac{3c^2}{4}z(t)\right\},$$

and using $a = c\cosh\alpha$, $b = c\sinh\alpha$, this result becomes

$$\Omega_1(t) = \frac{a^2}{3a^2+b^2}\{2a^2(1+\sigma)+b^2\}z(t)+\frac{a^2+\sigma(a^2+b^2)}{3(3a^2+b^2)}z^3(t)+\text{real const.} \quad (6.7)$$

This result agrees with the result given by Love that was mentioned in section 2 of this paper. From (4.4)

$$A'_{2,0}-\bar{A}_{2,0} = 0,$$

$$A_{2,1} = \tfrac{1}{8}ic^3[(1+2\sigma)\cosh^2\alpha-(1+\sigma)(2\cosh 2\alpha-1)],$$

$$A_{2,3} = \frac{ic^3}{24(2\cosh 2\alpha-1)}[(1+2\sigma)\cosh^2\alpha-(1+\sigma)],$$

$$A_{2,n} = 0 \quad (n \neq 0,1,3).$$

Thus $\Omega_2(t) = A_{2,0}+A_{2,1}(e^{\alpha}t+e^{-\alpha}t^{-1})+A_{2,3}(e^{3\alpha}t^3+e^{-3\alpha}t^{-3}),$

which as before reduces to

$$\Omega_2(t) = \frac{-ib^2}{a^2+3b^2}\{a^2+2b^2(1+\sigma)\}z(t)+\frac{i\{b^2+\sigma(a^2+b^2)\}}{3(a^2+3b^2)}z^3(t)+\text{real const.} \quad (6.8)$$

From (4.6),

$$A_{3,0}-\bar{A}_{3,0} = \tfrac{1}{2}ic^2\cosh 2\alpha, \qquad A_{3,2} = \tfrac{1}{2}ic^2\operatorname{sech} 2\alpha,$$

$$A_{3,n} = 0 \quad (n \neq 0,2),$$

so that $\Omega_3(t) = A_{3,0}+A_{3,2}(e^{2\alpha}t^2+e^{-2\alpha}t^{-2}),$

or

$$\Omega_3(t) = \frac{i(a^2-b^2)}{2(a^2+b^2)}z^2(t)+\text{const}, \quad (6.9)$$

which agrees with the well known result for the torsion function. From (5.1),

$$\Gamma = \tfrac{1}{2}\tau\pi\mu i\{4A_{3,2}(e^{2\alpha}b_{-2}-e^{-2\alpha}b_2)-i(c_1\bar{a}_1+c_{-1}\bar{a}_{-1})\},$$

that is,

$$\Gamma = \frac{\tau\pi\mu a^3b^3}{a^2+b^2}. \quad (6.10)$$

7. A family of cross-sections with a circular arc as internal cut

In order to proceed with this cross-section we must search first for a transformation of the concentric circle type. The one used here is derived from one given by Love (9).

The transformation is

$$z = z(t) = \frac{4a\cot\beta}{2\cot\beta-i(te^{\alpha}+t^{-1}e^{-\alpha})}, \quad (7.1)$$

and the cross-section is one whose internal cut is a circular arc of radius a, subtending an angle 4β at the point $(a, 0)$ in the z-plane, the centre of the arc E being at the point $(2a, 0)$.

We may rewrite (7.1) in the form

$$z = 2a\cos\beta\left\{\frac{ie^{-\alpha}}{ie^{-\alpha}+kt}+\frac{ie^{-\alpha}k}{t-ie^{-\alpha}k}\right\} \tag{7.2}$$

where $k = \tan\frac{1}{2}\beta < 1$ since $0 < 2\beta < \pi$.

Remembering now that the cross-section of the cylinder is one of the Riemann surfaces only and that we require $|t| = 1$ on the boundary, we

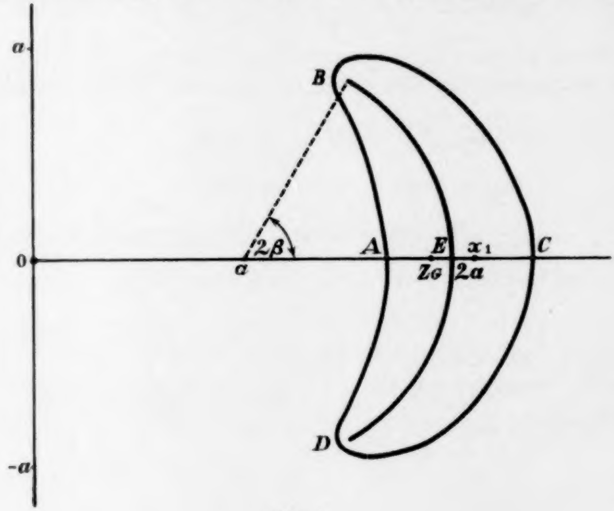

FIG. 1

choose, as in section 1, the particular Riemann surface corresponding to the ring-space $e^{-\alpha} < |t| < 1$ in the t-plane. This means that we are assuming that $\alpha > 0$ and that the radius of the *inner* circle corresponding to the slit is $e^{-\alpha}$. Thus, since $k < 1$ also, we have immediately $ke^{-\alpha} < 1$ and the second term in (7.2) can be expanded in powers of $(ike^{-\alpha}t^{-1})$.

If further we assume that $ke^{\alpha} < 1$, i.e. $k < e^{-\alpha}$, then the first term of (7.2) can be expanded in powers of $(ike^{\alpha}t)$ to give as the complete expansion

$$z = z(t) = \sum_{n=-\infty}^{\infty} a_n t^n,$$

where

$$a_n = 2a\cos\beta\, i^n k^n e^{n\alpha},$$

$$a_{-n} = 2a\cos\beta\, i^n k^n e^{-n\alpha}.$$

The particular form of this cross-section when $\beta = \frac{1}{6}\pi$, $e^{-\alpha} = \frac{3}{4}$, is shown in Fig. 1.

The point C of the figure corresponds to the value $t = e^{-\frac{1}{2}i\pi} = -i$ and provided the condition $e^{-\alpha} > k$ is satisfied then $|z-a| > a$ at this point;

so that $OC > 2a$. Also $OA < 2a$ under all conditions so that the condition $e^{-\alpha} > k$ ensures that the cross-section is the one required with the circular arc as an *internal cut*; otherwise part of the cut would lie outside the cross-section.

Next we wish to transfer the origin to the centroid of the cross-section, and using (3.2) we find

$$z_G = \frac{2a\cos\beta\{k^4(e^{2\alpha}-1)^2+e^{2\alpha}(1-k^2)(1-k^6)\}}{(1-k^4)(1-k^2e^{2\alpha})(e^{2\alpha}-k^2)}, \tag{7.3}$$

which is real, as was to be expected from symmetry.

Thus on transferring the origin to the centroid, the transformation becomes

$$z = z(t) = \sum_{n=-\infty}^{\infty} a_n t^n,$$

where $a_0 = 2a\cos\beta - z_G = 2ap\cos\beta,$

and $p = 1-\dfrac{z_G}{2a\cos\beta}$ is a real coefficient which is introduced for convenience; and for $n > 0$

$$\left.\begin{aligned} a_n &= 2a\cos\beta\, i^n k^n e^{n\alpha}, \\ a_{-n} &= 2a\cos\beta\, i^n k^n e^{-n\alpha}. \end{aligned}\right\} \tag{7.4}$$

All that is now required is the evaluation of the other coefficients b_n, c_n, etc. From (3.7),

$$b_0 = 4a^2\cos^2\beta\left[p^2+k^2\left\{\frac{e^{2\alpha}}{1-k^2e^{2\alpha}}+\frac{1}{e^{2\alpha}-k^2}\right\}\right]$$

and for $n > 0$

$$\left.\begin{aligned} b_n &= (-)^n b_{-n} = (-)^n \bar{b}_n \\ &= 4a^2\cos^2\beta[b'e^{n\alpha}+b''(-)^n e^{-n\alpha}]i^n k^n \end{aligned}\right\}, \tag{7.5}$$

where

$$b' = p+\frac{k^2e^{2\alpha}}{1-k^2e^{2\alpha}}-\frac{1}{1+e^{2\alpha}},$$

$$b'' = p+\frac{k^2}{e^{2\alpha}-k^2}-\frac{e^{2\alpha}}{1+e^{2\alpha}}.$$

From (3.8), $c_0 = 0$

and for $n > 0$ $\quad c_n = 8a^3\cos^3\beta[c'_n e^{n\alpha}+c''_n(-)^n e^{-n\alpha}]i^n k^n, \qquad (7.6)$

where

$$c'_n = b'\left[\frac{k^2}{(1+k^2)^2}+\frac{n(n-1)}{2}+\frac{nk^2e^{2\alpha}}{1-k^2e^{2\alpha}}+\frac{k^2e^{2\alpha}}{(1-k^2e^{2\alpha})^2}\right]+$$
$$+b''\left[\frac{e^{2\alpha}}{(1+e^{2\alpha})^2}-\frac{k^2}{(1+k^2)^2}-\frac{nk^2}{1+k^2}-\frac{n}{1+e^{2\alpha}}\right]+$$
$$+n\left[p^2+k^2\left\{\frac{e^{2\alpha}}{1-k^2e^{2\alpha}}+\frac{1}{e^{2\alpha}-k^2}\right\}\right],$$

$$c''_n = -b''\left[\frac{k^2e^{2\alpha}}{(e^{2\alpha}-k^2)^2}+\frac{e^{2\alpha}}{(1+e^{2\alpha})^2}\right].$$

Also for $n > 0$

$$c_{-n} = 8a^3\cos^3\beta[c'_{-n}(-)^n e^{n\alpha}+c''_{-n}e^{-n\alpha}]i^n k^n, \tag{7.7}$$

where

$$c'_{-n} = b'\left[\frac{k^2e^{2\alpha}}{(1-k^2e^{2\alpha})^2}+\frac{e^{2\alpha}}{(1+e^{2\alpha})^2}\right],$$

$$c''_{-n} = b'\left[\frac{k^2}{(1+k^2)^2}+\frac{nk^2}{1+k^2}+\frac{ne^{2\alpha}}{1+e^{2\alpha}}-\frac{e^{2\alpha}}{(1+e^{2\alpha})^2}\right]-$$

$$-b''\left[\frac{k^2}{(1+k^2)^2}+\frac{n(n-1)}{2}+\frac{nk^2}{e^{2\alpha}-k^2}+\frac{k^2e^{2\alpha}}{(e^{2\alpha}-k^2)^2}\right]-$$

$$-n\left[p^2+k^2\left\{\frac{e^{2\alpha}}{1-k^2e^{2\alpha}}+\frac{1}{e^{2\alpha}-k^2}\right\}\right].$$

From (3.12), (3.13),

$$\left.\begin{aligned} &g_0 = 0,\\ \text{and for } n>0\quad &g_n = -(-)^n g_{-n}\\ & = 8a^3\cos^3\beta[g'_n e^{n\alpha}+g''_n(-)^n e^{-n\alpha}]i^n k^n \end{aligned}\right\} \tag{7.8}$$

where

$$g'_n = \frac{n(n-3)}{2}+\frac{3n}{1+e^{2\alpha}}-\frac{3nk^2e^{2\alpha}}{1-k^2e^{2\alpha}}-\frac{3nk^2}{1+k^2}-\frac{6k^2}{(1+k^2)^2}-\frac{6e^{2\alpha}}{(1+e^{2\alpha})^2}+1,$$

$$g''_n = -\frac{n(n-3)}{2}-\frac{3ne^{2\alpha}}{1+e^{2\alpha}}+\frac{3nk^2}{e^{2\alpha}-k^2}+\frac{3nk^2}{1+k^2}+\frac{6k^2}{(1+k^2)^2}+\frac{6e^{2\alpha}}{(1+e^{2\alpha})^2}-1.$$

From (3.14), (3.15),

$$h_0 = 16a^3\cos^3\beta\left[2p\left\{2p^2-\frac{6k^2}{1+k^2}+\frac{3k^2e^{2\alpha}}{1-k^2e^{2\alpha}}+\frac{3k^2}{e^{2\alpha}-k^2}\right\}+\frac{6k^4}{(1+k^2)^2}+\right.$$

$$\left.+\frac{3k^4e^{4\alpha}}{(1-k^2e^{2\alpha})^2}+\frac{3k^4}{(e^{2\alpha}-k^2)^2}-\frac{6k^4e^{2\alpha}}{(1+k^2)(1-k^2e^{2\alpha})}-\frac{6k^4}{(1+k^2)(e^{2\alpha}-k^2)}\right],$$

and for $n > 0$

$$\begin{aligned} h_n &= (-)^n h_{-n}\\ &= 24a^3\cos^3\beta[h'_n e^{n\alpha}+h''_n(-)^n e^{-n\alpha}]i^n k^n, \end{aligned} \tag{7.9}$$

where

$$h'_n = 2p\left\{2p+(n-1)+\frac{2k^2e^{2\alpha}}{1-k^2e^{2\alpha}}-\frac{2k^2}{1+k^2}-\frac{2}{1+e^{2\alpha}}\right\}+\frac{nk^2e^{2\alpha}}{1-k^2e^{2\alpha}}-\frac{(n-2)k^2}{1+k^2}-$$

$$-\frac{(n-2)}{1+e^{2\alpha}}+\frac{2k^4}{(1+k^2)^2}+\frac{2k^4e^{4\alpha}}{(1-k^2e^{2\alpha})^2}-\frac{4k^2e^{2\alpha}}{(1+e^{2\alpha})(1-k^2e^{2\alpha})}+\frac{(n-1)(n-2)}{6},$$

$$h''_n = 2p\left\{2p+(n-1)+\frac{2k^2}{e^{2\alpha}-k^2}-\frac{2k^2}{1+k^2}-\frac{2e^{2\alpha}}{1+e^{2\alpha}}\right\}+\frac{nk^2}{e^{2\alpha}-k^2}-\frac{(n-2)k^2}{1+k^2}-$$

$$-\frac{(n-2)e^{2\alpha}}{1+e^{2\alpha}}+\frac{2k^4}{(1+k^2)^2}+\frac{2k^4}{(e^{2\alpha}-k^2)^2}-\frac{4k^2e^{2\alpha}}{(1+e^{2\alpha})(e^{2\alpha}-k^2)}+\frac{(n-1)(n-2)}{6}.$$

From (5.4), (5.5), for $n > 0$

$$l_n = e^{2n\alpha} l_{-n} = 24a^3 \cos^3\beta\, l'_n e^{n\alpha} i^n k^n,$$

where

$$l'_n = p\left\{p+(n-1)-\frac{2k^2}{1+k^2}\right\}-\frac{nk^2}{1+k^2}+\frac{2k^4}{(1+k^2)^2}+\frac{(n-1)(n-2)}{6}. \quad (7.10)$$

The expressions for the potential functions, torsion constant, and centre of flexure may now be summarized as follows:

$$\Omega_1(t) = \text{real const} + 8a^3\cos^3\beta \sum_{n=1}^{\infty} i^n k^n \{e^{n\alpha}-(-)^n e^{-n\alpha}\}^{-1}\{e^{n\alpha}t^n + e^{-n\alpha}t^{-n}\} \times$$
$$\times \left[\left(\frac{1+2\sigma}{12}\right)\{g'_n e^{n\alpha} + g''_n(-)^n e^{-n\alpha}\} + \right.$$
$$\left. + \left(\frac{1+\sigma}{n}\right)\{(c'_n + c'_{-n})e^{n\alpha} + (c''_n + c''_{-n})(-)^n e^{-n\alpha}\}\right],$$

$$\Omega_2(t) = \text{imaginary const} + 8a^3\cos^3\beta \sum_{n=1}^{\infty} i^{n+1} k^n \{e^{n\alpha}+(-)^n e^{-n\alpha}\}^{-1} \times$$
$$\times \{e^{n\alpha}t^n + e^{-n\alpha}t^{-n}\}\left[\frac{1+2\sigma}{4}\{h'_n e^{n\alpha} + h''_n(-)^n e^{-n\alpha}\} + \right.$$
$$\left. + \left(\frac{1+\sigma}{n}\right)\{(c'_n - c'_{-n})e^{n\alpha} + (c''_n - c''_{-n})(-)^n e^{-n\alpha}\}\right],$$

$$\Omega_3(t) = \text{imaginary const} - 4a^2 p \cos^2\beta \left\{\frac{k(e^{2\alpha}t^2+1) - 2ik^2 e^{\alpha} t}{(1-k^2)e^{\alpha}t - ik(e^{2\alpha}t^2+1)}\right\} +$$
$$+ 4a^2\cos^2\beta \sum_{n=1}^{\infty} i^{n+1}k^n\{e^{n\alpha}+(-)^n e^{-n\alpha}\}^{-1}\{e^{n\alpha}t^n + e^{-n\alpha}t^{-n}\} \times$$
$$\times \left\{\left(\frac{k^2 e^{2\alpha}}{1-k^2 e^{2\alpha}} - \frac{1}{1+e^{2\alpha}}\right)e^{n\alpha} + \left(\frac{k^2}{e^{2\alpha}-k^2} - \frac{e^{2\alpha}}{1+e^{2\alpha}}\right)(-)^n e^{-n\alpha}\right\}.$$

$$\Gamma = 8\tau\pi\mu a^4 \cos^4\beta \Big[\sum_{n=1}^{\infty} k^{2n}\{c'_n e^{2n\alpha} + c''_{-n}(-)^n e^{-2n\alpha} + c'_{-n} + (-)^n c''_n\} -$$
$$- \sum_{n=1}^{\infty} 2nk\{e^{n\alpha}-(-)^n e^{-n\alpha}\}\{e^{n\alpha}+(-)^n e^{-n\alpha}\}^{-1}\{b'e^{n\alpha} + b''(-)^n e^{-n\alpha}\}^2\Big],$$

while $y_0 = 0$ and

$$x_0 EB = -4\pi\mu a^5 \cos^5\beta [2(1+2\sigma)P - 8(1+\sigma)Q - 2(1+2\sigma)R + (1-2\sigma)T],$$

where

$$P = \sum_{n=1}^{\infty} nk^{2n}\{e^{n\alpha}-(-)^n e^{-n\alpha}\}\{e^{n\alpha}+(-)^n e^{-n\alpha}\}^{-1}\{b'e^{n\alpha}+b''(-)^n e^{-n\alpha}\} \times$$
$$\times \{h'_n e^{n\alpha} + h''_n(-)^n e^{-n\alpha}\},$$

$$Q = \sum_{n=1}^{\infty} k^{2n}\{e^{n\alpha}-(-)^n e^{-n\alpha}\}\{e^{n\alpha}+(-)^n e^{-n\alpha}\}^{-1}\{b'e^{n\alpha}+b''(-)^n e^{-n\alpha}\} \times$$
$$\times \{(c'_n - c'_{-n})e^{n\alpha} + (c''_n - c''_{-n})(-)^n e^{-n\alpha}\},$$

$$R = \sum_{n=1}^{\infty} nk^{2n} l'_n \{e^{n\alpha}-(-)^n e^{-n\alpha}\}\{b'e^{n\alpha}+b''(-)^n e^{-n\alpha}\},$$

$$T = \sum_{n=1}^{\infty} k^{2n}\{b'e^{n\alpha}+b''(-)^n e^{-n\alpha}\}\{(c'_n + c'_{-n})e^{n\alpha} + (c''_n + c''_{-n})(-)^n e^{-n\alpha}\}.$$

Writing, as usual, $E/\mu = 2(1+\sigma)$ and $\sigma' = \sigma/(1+\sigma)$, we can express x_0 in the form

$$x_0 = x_1+\sigma' x_2,$$

where x_1 gives the position of the centre of least strain. With the above expression for x_0 we have

$$x_1 B = -2\pi a^5 \cos^5\beta(2P-8Q-2R+T),$$
$$x_2 B = -2\pi a^5 \cos^5\beta(2P-2R-3T).$$

It is obvious that some of the terms in the above infinite series expressions may be summed to give finite expressions, as has been done in the simplest case of $\Omega_3(t)$. In general, however, finite expressions cannot be determined and the rapidity of the convergence of the series depends on the values of k and α. For the particular values of k and α used in section 7, viz. $k = 0{\cdot}26795$, $e^{\alpha} = \frac{4}{3}$ the series are rapidly convergent, and in particular the position of the centre of flexure is given by

$$x_1 = 0{\cdot}1723a, \qquad x_2 = 0{\cdot}0084a$$

referred to the centroid as origin. We note that x_2 is small in comparison with x_1 and the position of this centre of least strain, as well as the position of the centroid in this case, $z_G = 1{\cdot}9334a$, is marked on Fig. 1.

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STRESS CONCENTRATIONS IN A STRIP UNDER TENSION AND CONTAINING AN INFINITE ROW OF SEMICIRCULAR NOTCHES

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[Received 5 September 1957]

SUMMARY

The problem of determining the stresses in an infinite strip of finite breadth under tension and containing an infinite row of semicircular notches of equal radii placed at equal distances on the edges is studied theoretically. Stress concentration is calculated by a perturbation method in several cases, the radius of the notches and the distance between the centres of neighbouring notches being varied. The decrease in stress concentration is studied and compared with the results for an infinite strip under tension and containing two semicircular notches placed symmetrically on the edges. It is also compared with the stress concentrations in the similar problems of an infinite strip or a semi-infinite plate with notches under tension.

1. Introduction

In 1947 the problem of an infinite strip of finite breadth under tension and containing two semicircular notches placed symmetrically on the opposite edges was solved by Chih-Bing Ling (**1**). The most reliable determination of the stress concentration in this problem is that of M. Isida (**2**). The knowledge of the stress concentration produced by many notches is very important in practical applications, since the stress concentration in this case is smaller than the stress concentration in the case of few notches. It is possible, therefore, to improve a design by judiciously decreasing the weight of the part. In this paper, the stress concentration in an infinite strip under tension and containing an infinite row of semicircular notches of equal radii placed symmetrically at equal distances on the edges is obtained and compared with that given by M. Isida (**2**), in each case the radii of the notches and the distances between the notches being varied. The stress concentration is also compared with those obtained by the author (**3, 4**) for an infinite strip containing two pairs of semicircular notches placed symmetrically on the edges and for a semi-infinite plate containing an infinite row of semicircular notches under tension.

2. Two sets of functions

Consider an infinite strip of isotropic elastic material under the action of a longitudinal tension. Let the strip be bounded in the (x, y)-plane by lines equidistant from the x-axis. Let the edges of the strip be notched by an infinite number of pairs of equal semicircular notches, with the centres of any (arbitrary) pair of notches on the y-axis. Polar coordinates

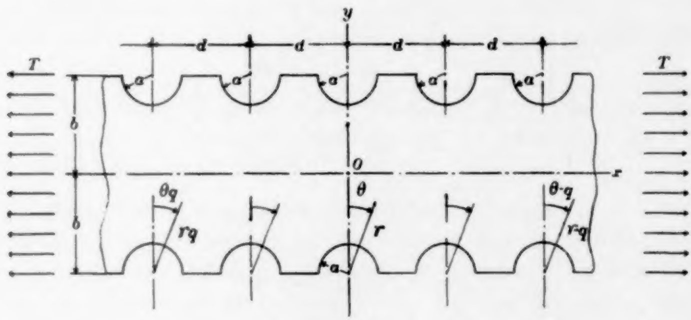

FIG. 1

(r, θ) referred to one of the centres of the notches on the y-axis as pole are taken as shown in Fig. 1. The width of the strip, the radius of the notch, and the distance between the centres of neighbouring notches are denoted by $2b$, a $(< b)$, and d respectively.

Following the method of Ling for solving the problem of an infinite strip of finite breadth under tension and containing two semicircular notches placed symmetrically on the opposite edges, we will first construct two series of biharmonic functions. These series of functions can be derived by differentiation from the stress functions for an unnotched strip under concentrated (*a*) normal, or (*b*) tangential force W acting at each of the infinite number of points $(0, \pm b)$, $(\pm d, \pm b)$, $(\pm 2d, \pm b)$,... on the straight edges. These two fundamental stress functions are as follows (**5**, **6**):

$$(a)\quad \chi_a = \frac{2W}{\pi}\times$$

$$\sum_{=-\infty}^{\infty}\int_0^{\infty}\left\{\frac{\sinh mb}{m^2}\,\frac{(1+mb\coth mb)\cosh my_q - my_q\sinh my_q}{\sinh 2mb+2mb}\cos mx_q - \frac{1}{2m^2}\right\}dm, \tag{1}$$

$$(b)\quad \chi_b = -\frac{2W}{\pi}\times$$

$$\times\sum_{q=-\infty}^{\infty}\int_0^{\infty}\frac{b\sinh mb\cosh my_q - y_q\cosh mb\sinh my_q}{m(\sinh 2mb+2mb)}\sin mx_q\,dm, \tag{2}$$

where
$$x_q = x+qd,\quad y_q = y,\quad x_0 = x,\quad y_0 = y. \tag{3}$$

We take as the series of biharmonic functions, which are even in both x and y, on account of symmetry, and which give no stresses on the straight edges or at infinity, the following:

$$\left.\begin{aligned}\chi_{2k} &= \frac{2\pi(-1)^{k+1}b^{2k+1}}{W(2k+1)!}\frac{\partial^{2k+2}\chi_a}{\partial x^{2k+2}}\\ \chi'_{2k} &= \frac{2\pi(-1)^{k+1}b^{2k}}{W(2k)!}\frac{\partial^{2k+1}\chi_b}{\partial x^{2k+1}}\end{aligned}\right\}, \tag{4}$$

where $k \geqslant 0$. To express these functions in terms of polar coordinates, we take, in addition to (r,θ) already defined, two sets of polar coordinates (r_q,θ_q) and (r_{-q},θ_{-q}), where $q > 0$, and refer to each of the infinite number of points $(-qd,-b)$ and $(qd,-b)$ as poles. Also, for brevity, we express the formulae in terms of dimensionless variables as follows:

$$\left.\begin{aligned}x/b = \xi,\quad (y+b)/b = \eta,\quad a/b = \lambda,\quad d/b = \beta\\ r/b = \rho,\quad r_q/b = \rho_q\end{aligned}\right\}. \tag{5}$$

Then each function, separated into two parts such that at the poles one still possesses the singularities but the other part does not, can be expressed as follows:

$$\left.\begin{aligned}\chi_{2k} &= \frac{2}{(2k+1)!}\int_0^{\infty}\biggl\{(1+u\eta)e^{-u\eta}+\\ &\quad+\frac{2u\sinh u\eta-2u^2\eta e^{-u\eta}+(1-e^{-2u})(\sinh u\eta-u\eta\cosh u\eta)}{\sinh 2u+2u}\biggr\}\times\\ &\qquad\times u^{2k}\cos u\xi\Bigl(1+2\sum_{q=1}^{\infty}\cos qu\beta\Bigr)\,du\\ \chi'_{2k} &= \frac{2}{(2k)!}\int_0^{\infty}\biggl\{\eta e^{-u\eta}+\frac{2\sinh u\eta-2u\eta e^{-u\eta}-\eta(1+e^{-2u})\sinh u\eta}{\sinh 2u+2u}\biggr\}\times\\ &\qquad\times u^{2k}\cos u\xi\Bigl(1+2\sum_{q=1}^{\infty}\cos qu\beta\Bigr)\,du\end{aligned}\right\}. \tag{6}$$

The relations between the coordinates are

$$\left.\begin{aligned}\zeta &= \xi+i\eta = \rho(\sin\theta+i\cos\theta)\\ \zeta_q &= \xi+q\beta+i\eta = \rho_q(\sin\theta_q+i\cos\theta_q)\quad (q=\pm1,\pm2,\ldots)\end{aligned}\right\}, \tag{7}$$

and therefore
$$\zeta_q = \zeta + q\beta. \tag{8}$$

The first part of the functions χ_{2k} and χ'_{2k} can be expressed in closed form with the aid of the integrals

$$\left.\begin{aligned}
&\int_0^\infty u^k e^{-u\eta}\cos u\xi\, du = \frac{k!}{\rho^{k+1}}\cos(k+1)\theta\\
&2\int_0^\infty u^k e^{-u\eta}\cos u\xi \sum_{q=1}^{\infty}\cos qu\beta\, du\\
&\qquad = \sum_{q=1}^{\infty}\left\{\frac{k!}{\rho_q^{k+1}}\cos(k+1)\theta_q + \frac{k!}{\rho_{-q}^{k+1}}\cos(k+1)\theta_{-q}\right\}
\end{aligned}\right\}. \tag{9}$$

Further, the results can be expressed in terms of ρ and θ by using (7) and (8). The use of the complex variable greatly simplifies the work and the final results are as follows:

$$\left.\begin{aligned}
&\rho_q^{-(2k+1)}\cos(2k+1)\theta_q + \rho_{-q}^{-(2k+1)}\cos(2k+1)\theta_{-q}\\
&\qquad = \sum_{p=0}^{\infty}(-1)^{k+p}\frac{2(2k+2p+1)!\,\rho^{2p+1}}{(2k)!\,(2p+1)!\,(q\beta)^{2k+2p+2}}\cos(2p+1)\theta\\
&\rho_q^{-(2k+2)}\cos(2k+2)\theta_q + \rho_{-q}^{-(2k+2)}\cos(2k+2)\theta_{-q}\\
&\qquad = \sum_{p=0}^{\infty}(-1)^{k+p+1}\frac{2(2k+2p+1)!\,\rho^{2p}}{(2k+1)!\,(2p)!\,(q\beta)^{2k+2p+2}}\cos 2p\theta \quad (q>0)
\end{aligned}\right\}. \tag{10}$$

For the second part the following expansions will be used:

$$\left.\begin{aligned}
\cosh u\eta\cos u\xi &= \sum_{n=0}^{\infty}\frac{(u\rho)^{2n}}{(2n)!}\cos 2n\theta\\
\sinh u\eta\cos u\xi &= \sum_{n=0}^{\infty}\frac{(u\rho)^{2n+1}}{(2n+1)!}\cos(2n+1)\theta
\end{aligned}\right\}. \tag{11}$$

Hence, omitting trivial terms which produce no effect on the stresses, we obtain the following expressions in polar coordinates:

$$\begin{aligned}
\chi_{2k} = {} & \frac{(2k+3)\cos(2k+1)\theta}{(2k+1)\rho^{2k+1}} + \frac{\cos(2k+3)\theta}{\rho^{2k+1}} - \\
& - \sum_{p=1}^{\infty}\frac{2p-1}{2p+1}\,{}^{2p}\alpha'_{2k}\,\rho^{2p+1}\cos(2p+1)\theta - \sum_{p=0}^{\infty}{}^{2p+2}\alpha'_{2k}\,\rho^{2p+3}\cos(2p+1)\theta + \\
& + \sum_{n=1}^{\infty}\left\{{}^{2n-1}\alpha_{2k}\,\rho^{2n}\cos 2n\theta - \frac{(2n-1)}{(2n+1)}\,{}^{2n}\alpha_{2k}\,\rho^{2n+1}\cos(2n+1)\theta\right\} + \\
& + \sum_{n=0}^{\infty}\{{}^{2n+1}\alpha_{2k}\,\rho^{2n+2}\cos 2n\theta - {}^{2n+2}\alpha_{2k}\,\rho^{2n+3}\cos(2n+1)\theta\},
\end{aligned} \tag{12}$$

$$\chi'_{2k} = \frac{\cos 2k\theta}{\rho^{2k}} + \frac{\cos(2k+2)\theta}{\rho^{2k}} + \sum_{p=1}^{\infty} {}^{2p-1}\beta'_{2k}\,\rho^{2p}\cos 2p\theta + \sum_{p=0}^{\infty} {}^{2p+1}\beta'_{2k}\,\rho^{2p+2}\cos 2p\theta$$
$$+ \sum_{n=1}^{\infty}\left\{{}^{2n-1}\beta_{2k}\,\rho^{2n}\cos 2n\theta - \frac{(2n-1)}{(2n+1)}\,{}^{2n}\beta_{2k}\,\rho^{2n+1}\cos(2n+1)\theta\right\} +$$
$$+ \sum_{n=0}^{\infty}\{{}^{2n+1}\beta_{2k}\,\rho^{2n+2}\cos 2n\theta - {}^{2n+2}\beta_{2k}\,\rho^{2n+3}\cos(2n+1)\theta\}, \tag{13}$$

where

$$\left.\begin{aligned}
{}^{2p}\alpha'_{2k} &= 2\sum_{q=1}^{\infty}(-1)^{k+p}\binom{2k+2p+1}{2k+1}\Big/(q\beta)^{2k+2p+2}\\
{}^{2n}\alpha_{2k} &= \binom{2k+2n+1}{2k+1}\left\{\frac{(2k+2n+2)I_{2k+2n+2}+I_{2k+2n+1}-J_{2k+2n+1}}{2^{2k+2n+1}}\right\}\\
{}^{2n-1}\alpha_{2k} &= \binom{2k+2n}{2k+1}\frac{(2k+2n+1)I_{2k+2n+1}}{2^{2k+2n}}\\
{}^{2p-1}\beta'_{2k} &= 2\sum_{q=1}^{\infty}(-1)^{k+p+1}\binom{2k+2p-1}{2k}\Big/(q\beta)^{2k+2p}\\
{}^{2n}\beta_{2k} &= \binom{2k+2n}{2k}\frac{(2k+2n+1)I_{2k+2n+1}}{2^{2k+2n}}\\
{}^{2n-1}\beta_{2k} &= \binom{2k+2n-1}{2k}\left\{\frac{(2k+2n)I_{2k+2n}-I_{2k+2n-1}-J_{2k+2n-1}}{2^{2k+2n-1}}\right\}
\end{aligned}\right\}, \tag{14}$$

and

$$\left.\begin{aligned}
I_k &= \sum_{q=1}^{\infty}\frac{2^k}{k!}\int_0^{\infty}\frac{u^k(1+2\cos qu\beta)}{\sinh 2u+2u}\,du\\
J_k &= \sum_{q=1}^{\infty}\frac{2^k}{k!}\int_0^{\infty}\frac{u^k e^{-2u}(1+2\cos qu\beta)}{\sinh 2u+2u}\,du
\end{aligned}\right\}. \tag{15}$$

By using Dirichlet's second integral (7), I_k and J_k may be written

$$\left.\begin{aligned}
I_k &= \frac{2^k}{k!}\int_0^{\infty}\frac{u^k\,du}{\sinh 2u+2u} +\\
&\qquad + \lim_{q\to\infty}\frac{2^{k+1}}{k!}\int_0^{\infty}\left[u^k\left\{\frac{\sin(q+\frac{1}{2})u\beta}{2\sin(\frac{1}{2}u\beta)}-\frac{1}{2}\right\}\Big/(\sinh 2u+2u)\right]du\\
&= \frac{2^{2k+1}}{k!\,\beta^{k+1}}\tfrac{1}{2}\pi\{f(0)+2f(\pi)+2f(2\pi)+\ldots\}\\
J_k &= \frac{2^{2k+1}}{k!\,\beta^{k+1}}\tfrac{1}{2}\pi\{f'(0)+2f'(\pi)+2f'(2\pi)+\ldots\}
\end{aligned}\right\}, \tag{16}$$

where $f(u) = \dfrac{u^k}{\sinh 4u/\beta + 4u/\beta}, \qquad f'(u) = \dfrac{u^k e^{-4u/\beta}}{\sinh 4u/\beta + 4u/\beta}.$ (17)

3. The stress function

When the strip is in a state of generalized plane stress, the stresses averaged across the thickness are derivable from a stress function χ which satisfies the biharmonic equation

$$\nabla^4 \chi = 0. \tag{18}$$

In the absence of the notches, a uniform tension T along the strip would be given by

$$\chi_0 = \tfrac{1}{4} b^2 T \rho^2 (1 + \cos 2\theta). \tag{19}$$

The method of satisfying the conditions when the notches are present is to add an infinite set of functions χ_{2k} and χ'_{2k} to χ_0 and then satisfy the boundary conditions at the rim of one of the notches by adjusting the parametric coefficients attached to the functions. Thus the required stress function may be constructed in the form

$$\chi = \chi_0 + b^2 T \sum_{k=0}^{\infty} \left(\frac{A_{2k+1}}{2k+2} \chi_{2k} + \frac{A_{2k}}{2k+1} \chi'_{2k} \right). \tag{20}$$

Expressing this in polar coordinates, we readily find

$$\begin{aligned} \chi/b^2 T = {} & \tfrac{1}{4} \rho^2 (1 + \cos 2\theta) + Q_0 \rho^2 + \\ & + \sum_{n=1}^{\infty} \frac{1}{(4n^2-1)} \left[\frac{(2n-1)A_{2n} + (2n+1)A_{2n-2}\rho^2}{\rho^{2n}} + \right. \\ & \qquad \left. + \{(2n+1)Q_{2n-2} + (2n-1)Q_{2n}\rho^2\}\rho^{2n} \right] \cos 2n\theta + \\ & + \sum_{n=0}^{\infty} \frac{1}{2n(2n+1)(2n+2)} \left[\frac{2n(2n+3)A_{2n+1} + (2n+1)(2n+2)A_{2n-1}\rho^2}{\rho^{2n+1}} - \right. \\ & \qquad \left. - \{(2n-1)(2n+2)Q_{2n-1} + 2n(2n+1)Q_{2n+1}\rho^2\}\rho^{2n+1} \right] \cos(2n+1)\theta, \end{aligned} \tag{21}$$

where

$$\left.\begin{aligned} Q_{2n} &= \sum_{k=0}^{\infty} (2n+1) \left\{ {}^{2n+1}\alpha_{2k} \frac{A_{2k+1}}{2k+2} + ({}^{2n+1}\beta_{2k} + {}^{2n+1}\beta'_{2k}) \frac{A_{2k}}{2k+1} \right\} \\ &= \sum_{k=0}^{\infty} (2n+1) \gamma_k^{2n} A_k \quad (n \geqslant 0) \\ Q_{2n-1} &= \sum_{k=0}^{\infty} 2n \left\{ ({}^{2n}\alpha_{2k} + {}^{2n}\alpha'_{2k}) \frac{A_{2k+1}}{2k+2} + {}^{2n}\beta_{2k} \frac{A_{2k}}{2k+1} \right\} \\ &= -\sum_{k=0}^{\infty} 2n \delta_k^{2n-1} A_k \quad (n > 0) \end{aligned}\right\}, \tag{22}$$

and

$$\left.\begin{aligned}
\gamma_{2k}^{2n} &= \binom{2k+2n+1}{2k}\left\{\frac{(2k+2n+2)I_{2k+2n+2}-I_{2k+2n+1}-J_{2k+2n+1}}{2^{2k+2n+1}}+\right.\\
&\qquad\left.+2\sum_{q=1}^{\infty}\frac{(-1)^{k+n}}{(q\beta)^{2k+2n+2}}\right\}\Big/(2k+1)\\
\gamma_{2k+1}^{2n} &= \binom{2k+2n+2}{2k+1}\frac{(2k+2n+3)I_{2k+2n+3}}{2^{2k+2n+2}}\Big/(2k+2)\\
\delta_{2k}^{2n-1} &= -\binom{2k+2n}{2k}\frac{(2k+2n+1)I_{2k+2n+1}}{2^{2k+2n}}\Big/(2k+1)\\
\delta_{2k+1}^{2n-1} &= -\binom{2k+2n+1}{2k+1}\left\{\frac{(2k+2n+2)I_{2k+2n+2}+I_{2k+2n+1}-J_{2k+2n+1}}{2^{2k+2n+1}}+\right.\\
&\qquad\left.+2\sum_{q=1}^{\infty}\frac{(-1)^{k+n}}{(q\beta)^{2k+2n+2}}\right\}\Big/(2k+2)
\end{aligned}\right\}. \tag{23}$$

A_k are parametric coefficients to be determined from the boundary conditions at the rim of the notch, that is

$$\sigma_r = 0, \quad \tau_{r\theta} = 0 \quad \text{at} \quad \rho = \lambda, \quad -\tfrac{1}{2}\pi \leqslant \theta \leqslant \tfrac{1}{2}\pi. \tag{24}$$

The expressions for the stresses are obtained by using (21) and are first expanded at $\rho = \lambda$ into Fourier series between $\theta = -\frac{1}{2}\pi$ and $\theta = \frac{1}{2}\pi$ as required on the rim. It is found that the resulting expansions contain terms of cosines or sines of even multiples of θ only. Equating the coefficient of each term to zero, we have

$$\tfrac{1}{2}+2Q_0-\tfrac{1}{2}f_0 = 0, \tag{25}$$

and for $n \geqslant 1$

$$\left.\begin{aligned}
&\tfrac{1}{2}\delta_{1,n}+(2n+2)A_{2n-2}\lambda^{-2n}+2nA_{2n}\lambda^{-2n-2}+2nQ_{2n-2}\lambda^{2n-2}+\\
&\qquad+(2n-2)Q_{2n}\lambda^{2n}+f_n = 0\\
&-\tfrac{1}{4}\delta_{1,n}+A_{2n-2}\lambda^{-2n}+A_{2n}\lambda^{-2n-2}-Q_{2n-2}\lambda^{2n-2}-Q_{2n}\lambda^{2n}+g_n = 0
\end{aligned}\right\}, \tag{26}$$

where $\delta_{m,n} = 1$ or 0 as m is equal to n or not, and

$$\left.\begin{aligned}
f_n &= \sum_{s=0}^{\infty}\{(2s+3)(A_{2s-1}\lambda^2+A_{2s+1})\lambda^{-2s-3}-\\
&\qquad-(2s-1)(Q_{2s-1}+Q_{2s+1}\lambda^2)\lambda^{2s-1}\}\frac{4(-1)^{n+s}(2s+1)}{\pi\{(2s+1)^2-4n^2\}}\\
g_n &= \sum_{s=0}^{\infty}\{(2s+1)A_{2s-1}\lambda^{-2s-1}+(2s+3)A_{2s+1}\lambda^{-2s-3}+\\
&\qquad+(2s-1)Q_{2s-1}\lambda^{2s-1}+(2s+1)Q_{2s+1}\lambda^{2s+1}\}\frac{4(-1)^{n+s}}{\pi\{(2s+1)^2-4n^2\}}
\end{aligned}\right\}. \tag{27}$$

The set of equations may be replaced by

$$A_{2n-2}\lambda^{-2n}+2nQ_{2n-2}\lambda^{2n-2}+(2n-1)Q_{2n}\lambda^{2n}+(2n-1)F_n+\tfrac{1}{2}\delta_{1,n}=0, \tag{28}$$

$$\left.\begin{aligned} F_0+2Q_0+\tfrac{1}{2} &= 0 \\ G_n+Q_{2n-2}\lambda^{2n-2}+2Q_{2n}\lambda^{2n}+Q_{2n+2}\lambda^{2n+2}+\tfrac{1}{4}\delta_{1,n} &= 0 \end{aligned}\right\}, \tag{29}$$

where

$$\left.\begin{aligned} F_n &= \sum_{s=1}^{\infty}({}^n\phi_{2s-2}A_{2s-1}\lambda^{-2s-1}-{}^n\mu_{2s}Q_{2s-1}\lambda^{2s-1}) \\ G_n &= \sum_{s=1}^{\infty}({}^n\psi_{2s-2}A_{2s-1}\lambda^{-2s-1}-{}^n\nu_{2s}Q_{2s-1}\lambda^{2s-1}) \end{aligned}\right\}, \tag{30}$$

and

$$\left.\begin{aligned} {}^n\phi_{2s-2} &= \frac{8(-1)^{n+s}(2s+1)}{\pi\{4s^2-(2n-1)^2\}(2n+2s+1)}, \qquad {}^n\psi_{2s-2} = \frac{-4}{2n+2s-1}\,{}^{n+1}\phi_{2s-2} \\ {}^n\mu_{2s} &= \frac{8(-1)^{n+s}(2s-1)}{\pi\{4s^2-(2n-1)^2\}(2n-2s+1)}, \qquad {}^n\nu_{2s} = \frac{-4}{2n-2s-1}\,{}^{n+1}\mu_{2s} \end{aligned}\right\}. \tag{31}$$

These equations for the biharmonic stress function and boundary conditions remain invariant even if the equations are expressed in terms of polar coordinates associated with any other one of the notches. Then, if the boundary conditions on one of the notches are satisfied, the corresponding conditions on the other notches are automatically satisfied.

A formal solution of equations (28) and (29) can be obtained as in Ling's paper (**1**). But, since this method is one of laborious successive approximations, a perturbation method, as in Isida's paper (**2**), in which λ is the perturbation parameter, is introduced here.

4. A perturbation method

In order to expand A_n into series of Q_k, we first require to solve the sets of equations for the $(A_{2n+1}\lambda^{-2n-3})$ given by (29) after setting $Q_k\lambda^k=1$, $Q_m\lambda^m=0$ $(m\neq k)$. Then substituting the given values of $(A_{2n+1}\lambda^{-2n-3})$ into (28), the values of $(A_{2n}\lambda^{-2n-2})$ are to be obtained. These calculations are not carried out here since Isida (**2**) gave the results, valid for this case, by solving the eight sets of equations obtained by setting $k=1$ to 8. When the values of $(A_n\lambda^{-n-2})$ are thus obtained, the values of $(A_n\lambda^{-n-2})$ can be determined as linear multiples of $Q_k\lambda^k$. The relations between A_n and Q_k are then expressed in the form

$$A_n = (\tfrac{1}{4}\lambda^{n+2})a_0^n+\sum_{k=0}^{\infty}(-1)^k\lambda^{n+k+2}\{a_k^n/(k+1)\}Q_k, \tag{32}$$

where the values of a_k^n are the same as shown in (**2**), p. 8, Table 1. Next,

assuming that A_n and Q_n can be expanded in powers of λ when λ is small, the expansions may be obtained from (22) and (32) as follows:

$$\left.\begin{aligned} A_n &= \sum_{p=0}^{\infty} A_n^{(n+p+2)}\lambda^{n+p+2} \\ Q_n &= \sum_{p=0}^{\infty} Q_n^{(p+2)}\lambda^{p+2} \end{aligned}\right\}. \tag{33}$$

Equating the terms of the same order of λ in both sides after substituting (33) into (22) and (32), we obtain

$$\left.\begin{aligned} Q_{2n}^{(p+2)} &= \sum_{k=0}^{p} (2n+1)\gamma_k^{2n} A_k^{(p+2)} \\ Q_{2n-1}^{(p+2)} &= -\sum_{k=0}^{p} 2n\delta_k^{2n-1} A_k^{(p+2)} \end{aligned}\right\}, \tag{34}$$

and

$$\left.\begin{aligned} A_n^{(n+2)} &= \tfrac{1}{4}a_0^n, \qquad A_n^{(n+3)} = 0 \\ A_n^{(n+p+4)} &= \sum_{k=0}^{p} (-1)^k\{a_k^n/(k+1)\}Q_k^{(p-k+2)} \end{aligned}\right\}. \tag{35}$$

Setting in turn $p = 0, 1, 2,...$ in (34) and (35), we can obtain the series of the coefficients of equations (33) as far as required. That is

$$\left.\begin{aligned} &A_n^{(n+2)} = \tfrac{1}{4}a_0^n, \qquad A_n^{(n+3)} = 0, \qquad Q_{2n}^{(2)} = (2n+1)\gamma_0^{2n}A_0^{(2)} \\ &Q_{2n-1}^{(2)} = -2n\delta_0^{2n-1}A_0^{(2)}, \qquad Q_{2n}^{(3)} = (2n+1)\gamma_1^{2n}A_1^{(3)} \\ &Q_{2n-1}^{(3)} = -2n\delta_1^{2n-1}A_1^{(3)}, \qquad A_n^{(n+4)} = a_0^n\, Q_0^{(2)} \\ &A_n^{(n+5)} = a_0^n\, Q_0^{(3)} - \tfrac{1}{2}a_1^n\, Q_1^{(2)}, \quad ... \end{aligned}\right\}. \tag{36}$$

Since $\sigma_r = 0$ at the rims of the notches, the maximum stress is given in the form

$$\sigma_{\max} = (\sigma_r + \sigma_\theta)_{\rho=\lambda,\theta=0} = \frac{1}{b^2}\left(\frac{1}{\rho^2}\frac{\partial^2\chi}{\partial\theta^2} + \frac{1}{\rho}\frac{\partial\chi}{\partial\rho} + \frac{\partial^2\chi}{\partial\rho^2}\right)_{\rho=\lambda,\theta=0}. \tag{37}$$

Substituting (33) into (37) determined by using (21) and (22), and then expanding in powers of λ, we obtain

$$\begin{aligned} \sigma_{\max}/T = 4\Big[\tfrac{1}{4} &- \sum_{n=0}^{\infty} A_n^{(n+2)} + \Big\{Q_0^{(2)} - \sum_{n=0}^{\infty} A_n^{(n+4)}\Big\}\lambda^2 + \\ &+ \Big\{Q_0^{(3)} - \sum_{n=0}^{\infty} A_n^{(n+5)} - Q_1^{(2)}\Big\}\lambda^3 + \Big\{Q_0^{(4)} - \sum_{n=0}^{\infty} A_n^{(n+6)} - Q_1^{(3)} + Q_2^{(2)}\Big\}\lambda^4 + \\ &+ \Big\{Q_0^{(5)} - \sum_{n=0}^{\infty} A_n^{(n+7)} - Q_1^{(4)} - Q_3^{(2)} + Q_2^{(3)}\Big\}\lambda^5 + ...\Big]. \end{aligned} \tag{38}$$

From (36) and (38) the numerical formula for $\sigma_{\max}/T$ can be obtained for any given value of β. When $\lambda = 0$, the results are equivalent to Maunsell's (8). When $\lambda \neq 0$, it is necessary to prove convergence of the series, though this is usually difficult. However, by comparison with Howland's

work (9) and from the agreement of the numerical results with experiment, convergence looks likely for $\lambda < \min(0{\cdot}5,\ 0{\cdot}25\beta)$. The numerical calculations are made in the lower parts of this range.

5. Numerical calculations and results

The numerical calculations have been carried out for the cases $\beta = 0{\cdot}5$, 1, $\frac{1}{2}\pi$, and 2. First an evaluation of the integrals I_k and J_k was performed in each case. Then the values of γ_{2k}^{2n}, γ_{2k+1}^{2n}, δ_{2k}^{2n-1}, and δ_{2k+1}^{2n-1} were determined by substituting the values of I_k and J_k into (23). Finally, $\sigma_{\max}/T$ was obtained by using (36) and (38). In each numerical equation, for the terms of λ^{10}, λ^{11}, and λ^{12}, $-\sum_{n=0}^{\infty} a_8^n Q_8^{(2)}/9$, $-\sum_{n=0}^{\infty} (a_8^n Q_8^{(3)}/9 - a_9^n Q_9^{(2)}/10)$, and $-\sum_{n=0}^{\infty} (a_8^n Q_8^{(4)}/9 - a_9^n Q_9^{(3)}/10 + a_{10}^n Q_{10}^{(2)}/11)$ were omitted respectively. But, since the terms omitted seem to be practically negligible, the following equations may give the results to an adequate degree of approximation. They are

$$\begin{aligned}\sigma_{\max}/T(\beta = 0{\cdot}5) = {} & 3{\cdot}065 - (9{\cdot}7330\times 10)\lambda^2 + (1{\cdot}3781\times 10^{-5})\lambda^3 + \\ & + (5{\cdot}9708\times 10^3)\lambda^4 + (8{\cdot}3745\times 10^{-4})\lambda^5 - (2{\cdot}8128\times 10^5)\lambda^6 + \\ & + (2{\cdot}2097\times 10^{-2})\lambda^7 + (1{\cdot}2386\times 10^7)\lambda^8 + (3{\cdot}7545\times 10^{-1})\lambda^9 - \\ & - (5{\cdot}2588\times 10^8)\lambda^{10} + (3{\cdot}3195\times 10)\lambda^{11} + (2{\cdot}1880\times 10^{10})\lambda^{12},\end{aligned} \tag{39}$$

$$\begin{aligned}\sigma_{\max}/T(\beta = 1) = {} & 3{\cdot}065 - (1{\cdot}6733\times 10)\lambda^2 + (2{\cdot}4696\times 10^{-1})\lambda^3 + \\ & + (2{\cdot}6955\times 10^2)\lambda^4 + (4{\cdot}8029)\lambda^5 - (2{\cdot}7816\times 10^3)\lambda^6 + \\ & + (3{\cdot}6025\times 10)\lambda^7 + (2{\cdot}6392\times 10^4)\lambda^8 + (1{\cdot}2558\times 10^2)\lambda^9 - \\ & - (2{\cdot}7031\times 10^5)\lambda^{10} + (6{\cdot}6733\times 10^2)\lambda^{11} + (2{\cdot}8017\times 10^6)\lambda^{12},\end{aligned} \tag{40}$$

$$\begin{aligned}\sigma_{\max}/T(\beta = \tfrac{1}{2}\pi) = {} & 3{\cdot}065 - (3{\cdot}9658)\lambda^2 + (3{\cdot}8920)\lambda^3 + (1{\cdot}6983\times 10)\lambda^4 + \\ & + (3{\cdot}7174\times 10)\lambda^5 - (1{\cdot}7500\times 10^2)\lambda^6 + (9{\cdot}0669\times 10)\lambda^7 + \\ & + (2{\cdot}8368\times 10^2)\lambda^8 - (1{\cdot}5480\times 10)\lambda^9 - (9{\cdot}3469\times 10^2)\lambda^{10} - \\ & - (1{\cdot}8312\times 10^3)\lambda^{11} + (8{\cdot}9066\times 10^3)\lambda^{12},\end{aligned} \tag{41}$$

$$\begin{aligned}\sigma_{\max}/T(\beta = 2) = {} & 3{\cdot}065 - (2{\cdot}2238)\lambda^2 + (8{\cdot}2007)\lambda^3 - (1{\cdot}6238\times 10)\lambda^4 + \\ & + (5{\cdot}1573\times 10)\lambda^5 - (7{\cdot}8374\times 10)\lambda^6 + (2{\cdot}3089\times 10)\lambda^7 + \\ & + (1{\cdot}5464\times 10^2)\lambda^8 - (4{\cdot}5787\times 10^2)\lambda^9 + (1{\cdot}1430\times 10^3)\lambda^{10} - \\ & - (2{\cdot}4943\times 10^3)\lambda^{11} + (3{\cdot}7532\times 10^3)\lambda^{12}.\end{aligned} \tag{42}$$

In Table 1, the values of $\sigma_{\max}/T$ calculated from the above equations are shown and are compared with those given by Isida (2) for Ling's problem.

TABLE 1

Values of σ_{max}/T

λ	$\beta = 0{\cdot}5$	$\beta = 1$	$\beta = \frac{1}{2}\pi$	$\beta = 2$	*M. Isida*
0·0	3·065	3·065	3·065	3·065	3·065
0·05	2·86	3·03	3·06	3·06	
0·075	2·67	2·98	3·04	3·06	
0·10		2·92	3·03	3·05	3·05
0·15		2·80	3·00	3·04	
0·20		2·70	2·97	3·03	3·04
0·25			2·95	3·03	3·04
0·30			2·95	3·03	3·05
0·35				3·06	
0·40					3·11

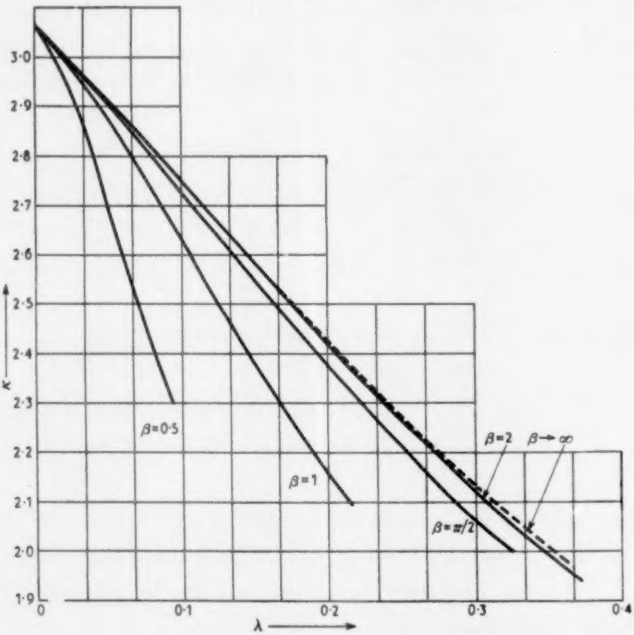

FIG. 2

In Fig. 2, the values of stress concentration factor $\kappa = \sigma_{max}(1-\lambda)/T$ calculated from the values in Table 1 are given. Here those given by Isida are shown in dotted line. In Fig. 3, the values of σ_{max}/T in Table 1 are compared with those calculated by the author **(4)** for a semi-infinite plate under tension and containing an infinite row of semicircular notches. The latter is shown in dotted line. In Fig. 4, the values of the stress

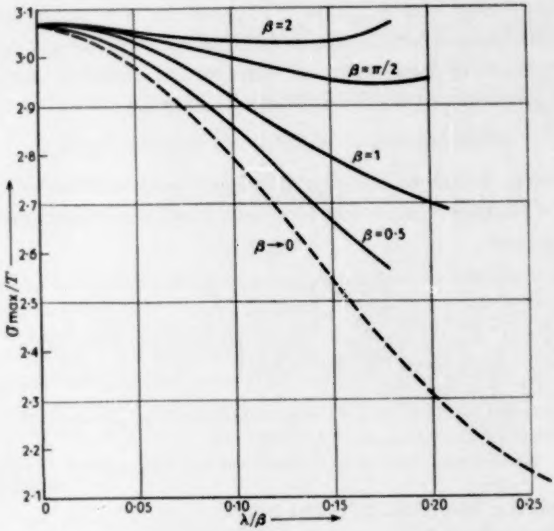
β=2
β=π/2
β=1
β=0·5
β→0
σmax/T
λ/β
3·1
3·0
2·9
2·8
2·7
2·6
2·5
2·4
2·3
2·2
2·1
0
0·05
0·10
0·15
0·20
0·25

FIG. 3

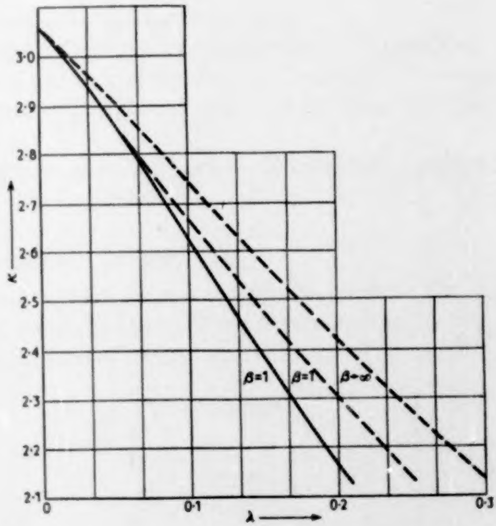
β=1
β=1
β→∞
K
λ
3·0
2·9
2·8
2·7
2·6
2·5
2·4
2·3
2·2
2·1
0
0·1
0·2
0·3

FIG. 4

concentration factor $\kappa = \sigma_{\max}(1-\lambda)/T$ given in this paper for $\beta = 1 \cdot 0$, those given by the author (**3**) for an infinite strip in the same conditions but only containing two pairs of semicircular notches, and those of M. Isida (**2**) are compared. The values of the last two are shown in dotted lines.

In conclusion, I wish to record my indebtedness to Professor Emeritus S. Higuchi of Tohoku University for much kind advice and many stimulating discussions.

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THE FLEXURAL VIBRATIONS OF A CUT THIN RING

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[Received 3 October 1957]

SUMMARY

There appears to be little published work on the flexural vibrations of a cut thin ring apart from a recent paper by H. Hasselgruber (**1**) who used the Rayleigh–Ritz method to find the frequency and shape of the first three modes. In the present paper these vibrations are investigated using exact methods, and tabulations are given for the first ten modes of symmetrical and anti-symmetrical vibration. There is no difficulty in extending the method to treat many problems of the flexural vibrations of circular arcs.

Nomenclature

We shall use the following notation:

T = circumferential tension.
Q = radial shearing force.
M = bending moment.
R = radius of the ring.
A = cross-sectional area of the ring.
w, v = radial and tangential displacements respectively.
m = mass per unit circumferential length of the ring.
k = radius of gyration of the cross-section of the ring.
t = time.
θ = angular coordinate, origin diametrically opposite the cut.
E = Young's modulus.

1. Fundamental equations

The coordinate system for the cut ring is shown in Fig. 1, and the forces acting on an elemental portion are shown in Fig. 2. For small displacements the equations of motion are

$$\left.\begin{aligned} \frac{\partial Q}{\partial \theta} - T &= mR\frac{\partial^2 w}{\partial t^2} \\ \frac{\partial T}{\partial \theta} + Q &= mR\frac{\partial^2 v}{\partial t^2} \\ \frac{\partial M}{\partial \theta} - QR &= mk^2\frac{\partial^2}{\partial t^2}\left(v - \frac{\partial w}{\partial \theta}\right) \end{aligned}\right\}, \tag{1}$$

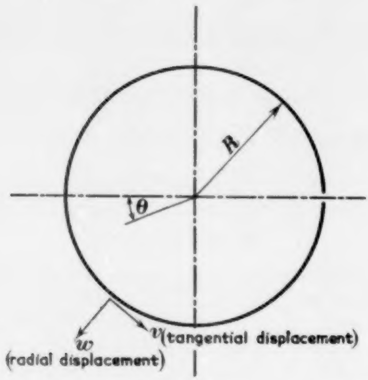

FIG. 1

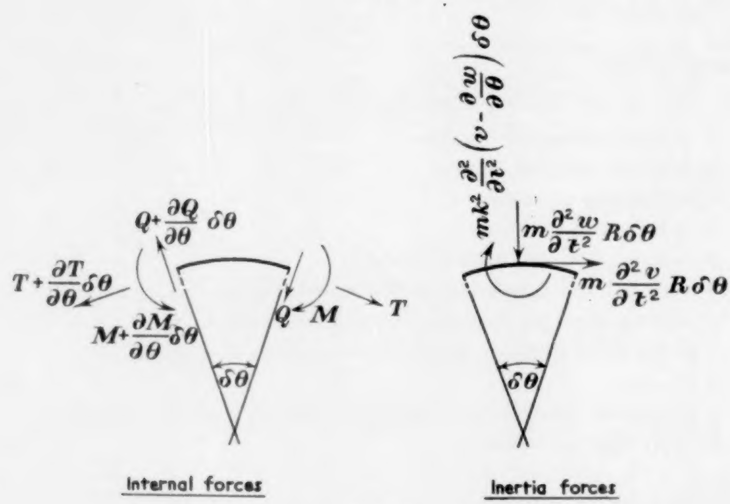

FIG. 2

the relationships between the strain and tension and bending moment being respectively

$$\left.\begin{aligned} T &= \frac{EA}{R}\left(\frac{\partial v}{\partial \theta}+w\right) \\ M &= \frac{EAk^2}{R^2}\frac{\partial}{\partial \theta}\left(v-\frac{\partial w}{\partial \theta}\right) \end{aligned}\right\}. \qquad (2)$$

Now, it is usual in this type of analysis to neglect the rotatory inertia effect, which amounts to neglecting the right-hand member of the third

of equations (1), and to assume that the central line of the ring is inextensible. From the first of equations (2) this implies that

$$w = -\frac{\partial v}{\partial \theta}. \tag{3}$$

Resulting from these considerations, the governing equation for the motion is found to be

$$\left\{\frac{\partial^2}{\partial\theta^2}\left(1+\frac{\partial^2}{\partial\theta^2}\right)^2 - \frac{R^4}{c^2k^2}\frac{\partial^2}{\partial t^2}\left(1-\frac{\partial^2}{\partial\theta^2}\right)\right\}M = 0, \tag{4}$$

where c is the velocity of wave propagation in a thin straight rod, i.e.

$$c = \sqrt{\frac{EA}{m}}.$$

2. Symmetrical vibrations

The normal functions for free vibration in the symmetrical modes are determined by taking the complete primitive of equation (4) to be of the form

$$M = \sum_{\kappa=1}^{3} a_\kappa \cos(\lambda_\kappa \theta)\cos(pt+\epsilon) \tag{5}$$

where the a_κ are, as yet, arbitrary constants, $p/(2\pi)$ is the frequency of vibration, and the λ_κ are the roots of the equation

$$\frac{R^4p^2}{c^2k^2} = \frac{\lambda^2(1-\lambda^2)^2}{1+\lambda^2}. \tag{6}$$

The corresponding expressions for the circumferential tension and radial shear are

$$T = -\frac{2}{R}\sum_{\kappa=1}^{3} a_\kappa \frac{\lambda_\kappa^2}{1+\lambda_\kappa^2}\cos(\lambda_\kappa \theta)\cos(pt+\epsilon), \tag{7}$$

$$Q = -\frac{1}{R}\sum_{\kappa=1}^{3} a_\kappa \lambda_\kappa \sin(\lambda_\kappa \theta)\cos(pt+\epsilon), \tag{8}$$

and for a complete record, the radial and tangential displacements are

$$w = -\frac{R^2}{EAk^2}\sum_{\kappa=1}^{3} a_\kappa \frac{1}{1-\lambda_\kappa^2}\cos(\lambda_\kappa \theta)\cos(pt+\epsilon) \tag{9}$$

and

$$v = \frac{R^2}{EAk^2}\sum_{\kappa=1}^{3} a_\kappa \frac{1}{\lambda_\kappa(1-\lambda_\kappa^2)}\sin(\lambda_\kappa \theta)\cos(pt+\epsilon). \tag{10}$$

The frequency $p/(2\pi)$ becomes determinate when the boundary conditions are satisfied and this requires the vanishing of the bending moment,

circumferential tension, and radial shear at $\theta = \pi$. This gives the determinantal equation

$$\begin{vmatrix} 1 & 1 & 1 \\ \lambda_1(\lambda_1^2-1)^2\tan\lambda_1\pi & \lambda_2(\lambda_2^2-1)^2\tan\lambda_2\pi & \lambda_3(\lambda_3^2-1)^2\tan\lambda_3\pi \\ (\lambda_1^2-1)^2 & (\lambda_2^2-1)^2 & (\lambda_3^2-1)^2 \end{vmatrix} = 0 \qquad (11)$$

which is valid provided that there are no equal roots λ.

In the evaluation of the frequencies from equations (6) and (11), it is soon found that there are advantages in using λ_1 as the independent variable, so that λ_2, λ_3, and p are dependent variables. Now, equation (6) is a cubic in λ^2 and we may say that λ_1^2 must always be real and, furthermore, lies outside the range $-1 \leqslant \lambda_1^2 \leqslant 0$ for real and non-trivial p. Again, from this equation, it is readily deduced that

$$\left.\begin{aligned} \lambda_1^2+\lambda_2^2+\lambda_3^2 &= 2 \\ \lambda_1^2\lambda_2^2+\lambda_2^2\lambda_3^2+\lambda_3^2\lambda_1^2 &= 1-\frac{R^4p^2}{c^2k^2} \\ \lambda_1^2\lambda_2^2\lambda_3^2 &= \frac{R^4p^2}{c^2k^2} \end{aligned}\right\}, \qquad (12)$$

so that $$\lambda_2^2,\lambda_3^2 = \frac{2-\lambda_1^2}{2}\left[1\pm\left\{1-\frac{4(1-\lambda_1^2)^2}{(1+\lambda_1^2)(2-\lambda_1^2)^2}\right\}^{\frac{1}{2}}\right], \qquad (13)$$

and from this last it is seen that if $\lambda_1^2 < -1$, then either λ_2^2 or λ_3^2 is positive. Hence, there is no loss in generality by considering only values

$$\lambda_1^2 > 0, \qquad (14)$$

and it is worth noting that when $0 < \lambda_1^2 < \frac{1}{2}(7+\sqrt{17})$, the roots λ_2^2 and λ_3^2 are complex and an approximation to λ_1 gives a purely imaginary value to the determinant of equation (11).

Expansion of equation (13) yields

$$\left.\begin{aligned} \lambda_2^2 &= -\lambda_1^2+3+\frac{4}{\lambda_1^4}\cdots \\ \lambda_3^2 &= -1-\frac{4}{\lambda_1^4}\cdots \end{aligned}\right\}, \qquad (15)$$

and so the determinantal equation (11) reduces to

$$\lambda_1\tan\lambda_1\pi \doteqdot \lambda_2\tan\lambda_2\pi \qquad (16)$$

when terms of order λ_1^{-4} are neglected in comparison with unity. Furthermore, substituting the first of equations (15) into this last, it is found that

$$\tan\lambda_1\pi \doteqdot -1+\frac{3}{2\lambda_1^2}$$

or, in other words, $$\lambda_1 \doteqdot (n-\tfrac{1}{4})+\frac{3}{4\pi(n-\frac{1}{4})^2} \qquad (17)$$

where n is an integer. This formula improves in accuracy as the integer n increases. The values of λ_1, λ_2, λ_3, and R^2p/ck have been calculated for the first ten modes and are appended in the table below.

TABLE 1

Mode	λ_1†	λ_2	λ_3	R^2p/ck
1	1·24927	0·53387+0·25559i	0·53387−0·25559i	0·43767‡
2	1·85206	0·46878+0·96686i	0·46878−0·96686i	2·13835‡
3	2·78662	2·15403i	1·06084i	6·36764
4	3·76855	3·34307i	1·01285i	12·7604
5	4·76122	4·43397i	1·00452i	21·2065
6	5·75752	5·49045i	1·00201i	31·6749
7	6·75540	6·52942i	1·00103i	44·1543
8	7·75407	7·55807i	1·00058i	58·6399
9	8·75318	8·58006i	1·00035i	75·1294
10	9·75255	9·59749i	1·00023i	93·6213

† These values of λ_1 were calculated by Dr. K. N. Dodd, Royal Aircraft Establishment.
‡ Hasselgruber (1) calculated these values as 0·4377 and 2·1856 using the Rayleigh–Ritz method.

The values of the arbitrary constants a_κ in equation (5) are given by

$$\left.\begin{aligned} a_1 &= \beta(\lambda_2^2-\lambda_3^2)(1+\lambda_1^2)\sec\lambda_1\pi \\ a_2 &= \beta(\lambda_3^2-\lambda_1^2)(1+\lambda_2^2)\sec\lambda_2\pi \\ a_3 &= \beta(\lambda_1^2-\lambda_2^2)(1+\lambda_3^2)\sec\lambda_3\pi \end{aligned}\right\}, \tag{18}$$

where β is a normalizing constant such that

$$\int_{-\pi}^{\pi} M^2\,d\theta = \cos^2(pt+\epsilon). \tag{19}$$

It is worth noting that since the bending is the only contribution to the potential energy then

$$\int_{-\pi}^{\pi} M_n M_m\,d\theta = 0 \tag{20}$$

for any mode n not identical with mode m. The calculated values of the normalized constants a_κ are given in the table below.

TABLE 2

Mode	a_1	a_2	a_3
1	0·2612	0·2019+0·1382i	0·2109−0·1382i
2	−0·5256	0·004233+0·02227i	0·004233−0·02227i
3	0·5557	$0·1053\times10^{-2}$	$-0·1568\times10^{-2}$
4	−0·5612	$0·2313\times10^{-4}$	$-0·1477\times10^{-3}$
5	0·5628	$0·7352\times10^{-6}$	$-0·3041\times10^{-4}$
6	−0·5634	$0·2633\times10^{-7}$	$-0·8934\times10^{-5}$
7	0·5637	$0·1001\times10^{-8}$	$-0·3264\times10^{-5}$
8	−0·5639	$0·3937\times10^{-10}$	$-0·1386\times10^{-5}$
9	0·5640	$0·1584\times10^{-11}$	$-0·6565\times10^{-6}$
10	−0·5640	$0·6467\times10^{-13}$	$-0·3384\times10^{-6}$

3. Antisymmetrical vibrations

The analysis for the antisymmetrical vibrations follows precisely the same lines as just discussed and so a catalogue of results will suffice.

The bending moment, circumferential tension, and radial shear are given by the expressions

$$M = \sum_{\kappa=1}^{3} a_\kappa \sin(\lambda_\kappa \theta)\cos(pt+\epsilon), \tag{21}$$

$$T = -\frac{2}{R}\sum_{\kappa=1}^{3} a_\kappa \frac{\lambda_\kappa^2}{1+\lambda_\kappa^2}\sin(\lambda_\kappa \theta)\cos(pt+\epsilon), \tag{22}$$

$$Q = \frac{1}{R}\sum_{\kappa=1}^{3} a_\kappa \lambda_\kappa \cos(\lambda_\kappa \theta)\cos(pt+\epsilon), \tag{23}$$

and the radial and tangential displacements by

$$w = -\frac{R^2}{EAk^2}\sum_{\kappa=1}^{3} a_\kappa \frac{1}{1-\lambda_\kappa^2}\sin(\lambda_\kappa \theta)\cos(pt+\epsilon), \tag{24}$$

$$v = -\frac{R^2}{EAk^2}\sum_{\kappa=1}^{3} a_\kappa \frac{1}{\lambda_\kappa(1-\lambda_\kappa^2)}\cos(\lambda_\kappa \theta)\cos(pt+\epsilon). \tag{25}$$

The determinantal equation for the evaluation of the frequencies $p/(2\pi)$ is

$$\begin{vmatrix} 1 & 1 & 1 \\ \lambda_1(\lambda_1^2-1)^2\cot\lambda_1\pi & \lambda_2(\lambda_2^2-1)^2\cot\lambda_2\pi & \lambda_3(\lambda_3^2-1)^2\cot\lambda_3\pi \\ (\lambda_1^2-1)^2 & (\lambda_2^2-1)^2 & (\lambda_3^2-1)^2 \end{vmatrix} = 0 \tag{26}$$

which is valid provided that there are no equal roots λ. The same remarks apply for the evaluation of the frequencies except that the determinantal equation (26) reduces now to

$$\lambda_1 \cot\lambda_1\pi \doteqdot \lambda_2 \cot\lambda_2\pi \tag{27}$$

when terms of order λ_1^{-4} are neglected in comparison with unity. The approximation for the root λ_1 is now

$$\lambda_1 \doteqdot (n+\tfrac{1}{4}) + \frac{3}{4\pi(n-\frac{1}{4})^2}, \tag{28}$$

where n is again an integer. This formula improves in accuracy as the

integer n increases. The values of λ_1, λ_2, λ_3 and R^2p/ck have been calculated for the first ten modes and are appended in the table below.

TABLE 3

Mode	λ_1†	λ_2	λ_3	R^2p/ck
1	1·46556‡	0·53527+0·60037i	0·53527−0·60037i	0·94816§
2	2·30773	0·16891+1·30052i	0·16891−1·30052i	3·96902
3	3·27532	2·77067i	1·02522i	9·30373
4	4·26419	3·89469i	1·00733i	16·7295
5	5·25909	4·96509i	1·00295i	26·1888
6	6·25633	6·01156i	1·00142i	37·6636
7	7·25466	7·04476i	1·00077i	51·1465
8	8·25358	8·06974i	1·00045i	66·6343
9	9·25284	9·08925i	1·00028i	84·1251
10	10·25230	10·10492i	1·00019i	103·618

† These values of λ_1 were calculated by Dr. K. N. Dodd, Royal Aircraft Establishment.

‡ This value is accurate only to within $\pm 0{\cdot}00002$ and λ_2, λ_3 and R^2p/ck have been calculated from this value.

§ Hasselgruber (1) calculated this value as 0·9709.

The arbitrary constants a_κ in equations (21) to (23) are given by

$$a_1 = (\lambda_2^2-\lambda_3^2)(1+\lambda_1^2)\operatorname{cosec}\lambda_1\pi,$$
$$a_2 = (\lambda_3^2-\lambda_1^2)(1+\lambda_2^2)\operatorname{cosec}\lambda_2\pi,$$
$$a_3 = (\lambda_1^2-\lambda_2^2)(1+\lambda_3^2)\operatorname{cosec}\lambda_3\pi. \tag{29}$$

It has, unfortunately, not been possible to tabulate the values of the normalized a_κ for these antisymmetrical vibrations.

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TRANSVERSE WAVES IN ELASTIC PLATES

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[Received 26 September 1957]

SUMMARY

Expressions are derived for the phase and group velocities of plane rotational waves in elastic plates with displacement vector parallel to the surfaces of the plates and perpendicular to the direction of propagation. It is shown that in a double plate two types of transverse wave can travel along the plate, each type consisting of an infinite number of modes. The behaviour of the waves is illustrated by dispersion curves, computed for the lowest modes. The paper concludes with an investigation of the properties of Love waves in a surface layer, including a discussion of the behaviour of the minimum group velocity, discovered by Jeffreys (**1**), for which a lower bound is found.

1. Introduction

Recent work on the propagation of vibrations in elastic plates has been almost confined to the study of displacements in planes parallel to the direction of propagation and at right angles to the faces of the plates. Whilst it is true that most of the energy of a disturbance will be transmitted by waves of this form, a proportion will, in general, travel as waves of rotation about axes perpendicular to the faces of the plates. Some properties of waves of this second kind were investigated by Love (**2**), Jeffreys (**3**), and Stoneley (**4**), in connexion with the propagation of waves on the surface of the earth, but their presence in plates has been neglected so far.

If it is assumed that at a large distance from a disturbance waves of this kind travelling along a plate have plane fronts, their displacements will be parallel to the faces of the plates and transverse to the direction of propagation. It will also be assumed that these waves can be resolved into harmonic components so that with the use of harmonic analysis results of a very general nature are obtained which are applicable to most kinds of initial value problems.

2. The equations of motion and the boundary conditions

A rectangular cartesian system of coordinates $O(x,y,z)$ is chosen so that the faces of the plates are planes $z = \text{const}$, and (u,v,w) denote the x, y, z components of the displacement vector $\mathbf{u}$. In the case of a plane wave travelling in the direction of the x-axis the z-component of the rotation $\nabla \wedge \mathbf{u}$,

$$\psi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{2.1}$$

will correspond to a displacement in the plane of the faces of the plate and perpendicular to the direction of propagation. The equation of motion is

$$c_1^2 \nabla^2 \psi = \frac{\partial^2 \psi}{\partial t^2}, \tag{2.2}$$

where $c_1 = (\mu/\rho)^{\frac{1}{2}}$ is the velocity of rotational waves, ρ is the density, and μ is Lamé's constant.

The conditions to be satisfied on a stress-free face are

$$\widehat{zz} = \widehat{zx} = \widehat{zy} = 0.$$

From the stress-strain relations we find that the condition on ψ at a stress-free face $z =$ const is

$$\frac{\partial \psi}{\partial z} = 0. \tag{2.3}$$

If the plane $z =$ const is the boundary between two media with different elastic properties then the stresses and displacements must be continuous across that face. This means that ψ and $\mu(\partial\psi/\partial z)$ are continuous across the boundary.

For the remainder of this paper we confine our attention to plane waves, v being the only component of the displacement considered.

3. Transverse waves in a plate

The plate is assumed to occupy the space between the planes $z = \mp h$ with boundary conditions

$$\frac{\partial \psi}{\partial z} = 0 \quad \text{on } z = \mp h. \tag{3.1}$$

Let ψ be composed of harmonic components of the form

$$\psi = \Psi e^{i(px-nt)}, \tag{3.2}$$

representing a plane wave travelling in the direction of the x-axis. Ψ is a function of z only, $n/2\pi$ is the frequency, and $\lambda = 2\pi/p$ is the wavelength of the waves. Substituting for ψ in the equation (2.2) gives

$$\frac{d^2\Psi}{dz^2} + \left(\frac{n^2}{c_1^2} - p^2\right)\Psi = 0, \tag{3.3}$$

where c_1 is the velocity of rotational waves in the medium. The general solution of the differential equation (3.3) is

$$\Psi = A \cos rz + B \sin rz, \tag{3.4}$$

where
$$r^2 = n^2/c_1^2 - p^2. \tag{3.5}$$

Thus, using the boundary conditions (3.1) it is seen that the only possibilities are that either $A = 0$, $rh = (m+\frac{1}{2})\pi$, or $B = 0$ and $rh = m\pi$,

where m is any positive integer. If $A = 0$ the displacements will be antisymmetrical about the plane $z = 0$, while in the case $B = 0$ the displacements will be symmetrical. We shall devote our attention to the antisymmetrical case, results for the symmetrical case being entirely analogous. For antisymmetrical displacements

$$\psi = Be^{i(px-nt)}\sin r_m z, \tag{3.6}$$

where
$$r_m = (m+\tfrac{1}{2})\pi/h \tag{3.7}$$

and
$$n^2/c_1^2-p^2 = r_m^2. \tag{3.8}$$

Thus ψ consists of an infinite number of branches, or modes, depending on the choice of the integer m. The branch with $m = 0$ will be called the first mode, that with $m = 1$ the second, and so on. From the equation (3.8) it is clear that

$$n^2 \geqslant (c_1 r_m)^2, \tag{3.9}$$

so that the minimum, or 'cut off', frequency in the $(m+1)$th mode is given by

$$n = (m+\tfrac{1}{2})\pi c_1/h. \tag{3.10}$$

Thus we see that no wave of this type can be propagated for which $n \leqslant c_1/2h$.

Writing c $(= n/p)$ for the phase velocity, we see from (3.8) that

$$c = c_1(1+r_m^2/p^2)^{\frac{1}{2}}. \tag{3.11}$$

As the ratio p/r_m varies from zero to infinity, v varies from infinity to the limit c_1 for waves which are very short compared with the thickness of the plate.

The group velocity V $(= dn/dp)$ is found from (3.8) to be given by

$$V = c_1(1+r_m^2/p^2)^{-\frac{1}{2}}. \tag{3.12}$$

Thus for short waves p/r_m is large and V will tend to the upper limit c_1, while for long waves p/r_m is small and V tends to zero.

It is beyond the scope of this paper to discuss at length the concept of group velocity and good accounts may be found in Jeffreys and Jeffreys (**5**) or Stratton (**6**). The disturbance associated with a particular wavelength will travel with the group velocity, while individual waves will travel with the phase velocity, but will change their periods, lengths, and velocities as they travel. In view of the simplicity of the expression for the group velocity in (3.12) it is possible to discover what happens to a wave after a long time. The total disturbance travelling as transverse waves may be expressed in the form

$$v(x,z,t) = \sum_{m=0}^{\infty} \sin r_m z \int_{-\infty}^{\infty} A_m(p)e^{i(px-nt)}\,dp, \tag{3.13}$$

where $A_m(p)$ depends only on the initial conditions at $t = 0$. Sufficient conditions for v and ψ to exist in this integral form are that

$$A_m(p) = O(1/mp^2), \tag{3.14}$$

as m or p tends to infinity. The integral in (3.13) can be readily evaluated for large x and t by the method of stationary phase, which yields the asymptotic expression

$$v(x,z,t) \sim \frac{(2\pi)^{\frac{1}{2}}c_1 t}{\xi^{\frac{3}{4}}} \sum_{m=0}^{\infty} r_m^{\frac{1}{2}} \sin r_m z A_m\left(\frac{xr_m}{\xi^{\frac{1}{2}}}\right) e^{-i(\xi^{\frac{1}{2}}r_m+\frac{1}{4}\pi)}, \tag{3.15}$$

where $\xi = c_1^2 t^2 - x^2$. Remembering to take the correct branch of $\xi^{\frac{1}{2}}$, it may be seen from the condition (3.14) that when $x = c_1 t$, v is zero, while if $x > c_1 t$, p becomes imaginary and the exponential term in (3.15) tends rapidly to zero. Thus it follows that the maximum velocity of propagation of the disturbance is c_1, and that the shorter waves will arrive shortly after the first signs of the disturbance, while the longer waves will suffer considerable dispersion and will travel more slowly. In addition, it can be shown that for a fixed wavelength the lower modes will suffer less dispersion and will travel more rapidly than the modes for which m is large.

4. Transverse waves in a double plate

We shall assume that the plate consists of two different materials, one, in which the velocity of rotational waves is c_1, occupying the space $0 < z < h$, while the other, in which the velocity of rotational waves is c_2, occupies the space $-h < z < 0$. There is no loss of generality in assuming that c_1 is greater than c_2. Here μ_1, μ_2 are the respective Lamé's constants for the two materials, and τ is the ratio μ_1/μ_2.

If ψ_1 and ψ_2 are the respective displacement functions in the two layers then the equations of motion are

$$\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial z^2} = \frac{1}{c_1^2}\frac{\partial^2\psi_1}{\partial t^2} \quad \text{when } 0 < z < h, \tag{4.1}$$

and

$$\frac{\partial^2\psi_2}{\partial x^2} + \frac{\partial^2\psi_2}{\partial z^2} = \frac{1}{c_2^2}\frac{\partial^2\psi_2}{\partial t^2} \quad \text{when } -h < z < 0. \tag{4.2}$$

The boundary conditions are

$$\frac{\partial\psi_1}{\partial z} = 0 \quad \text{on } z = h, \tag{4.3}$$

$$\frac{\partial\psi_2}{\partial z} = 0 \quad \text{on } z = -h, \tag{4.4}$$

and

$$\psi_1 = \psi_2, \quad \mu_1(\partial\psi_1/\partial z) = \mu_2(\partial\psi_2/\partial z) \quad \text{on } z = 0. \tag{4.5}$$

We assume that $$\psi_1 = \Psi_1 e^{i(px-nt)}, \tag{4.6}$$
and that $$\psi_2 = \Psi_2 e^{i(px-nt)}. \tag{4.7}$$
Substituting ψ_1 and ψ_2 from (4.6) and (4.7) in the equations of motion gives the equations
$$\frac{d^2\Psi_1}{dz^2}+\left(\frac{n^2}{c_1^2}-p^2\right)\Psi_1 = 0, \tag{4.8}$$
and
$$\frac{d^2\Psi_2}{dz^2}+\left(\frac{n^2}{c_2^2}-p^2\right)\Psi_2 = 0. \tag{4.9}$$
If $c = n/p$ is the phase velocity then there are three possibilities. Either (i) $c \geqslant c_1$, or (ii) $c_1 \geqslant c \geqslant c_2$, or (iii) $c_2 \geqslant c$.

Stoneley (7) treated the problem of a double plate as a limiting case of a double surface layer, but did not consider the possibility $c_1 > c$. He therefore erroneously concluded that the wave velocity in a double plate is greater than the velocity of rotational waves in either of the two layers.

(i) Taking $c \geqslant c_1$ we put
$$r_1^2 = n^2/c_1^2-p^2, \quad \text{and} \quad r_2^2 = n^2/c_2^2-p^2, \tag{4.10}$$
where r_1 and r_2 are real positive numbers. In this case the general solutions of the differential equations (4.8) and (4.9) are
$$\Psi_1 = A\cos r_1 z+B\sin r_1 z, \tag{4.11}$$
and
$$\Psi_2 = C\cos r_2 z+D\sin r_2 z, \tag{4.12}$$
where A, B, C, and D are constants. If these expressions are substituted for ψ_1 and ψ_2 in the boundary conditions (4.3) to (4.5) and the arbitrary constants are eliminated we obtain the transcendental equation
$$\tau r_1 \tan r_1 h = -r_2 \tan r_2 h. \tag{4.13}$$
This may be taken as an equation connecting the dependent variable n and the independent variable p, with the solution for n consisting of an infinite number of branches corresponding to modes of vibration. Since $c_1 > c_2$ it follows from (4.10) that $r_2 > r_1$. If r_1 lies in the interval
$$0 \leqslant hr_1 \leqslant \tfrac{1}{2}\pi,$$
then it is evident from (4.13) that r_2 is at least $\frac{1}{2}h\pi$. Thus we see from (4.10) that the minimum frequency of this type of wave is given by $n \geqslant c_2/2h$.

The phase velocity c is obtained from the equations (4.10), which yield
$$c^2 = \gamma c_2^2(r_2^2-r_1^2)/(r_2^2-\gamma r_1^2), \tag{4.14}$$
where $\gamma = c_1^2/c_2^2$. Differentiating the equations (4.10) and (4.13) with respect to p we obtain the group velocity V in the form
$$V = \gamma c_2^2(\phi r_1+r_2)/c(\gamma\phi r_1+r_2), \tag{4.15}$$

where $$\tau\phi = (hr_2\sec^2 hr_2+\tan hr_2)/(hr_1\sec^2 hr_1+\tan hr_1). \tag{4.16}$$

The shape of the dispersion curves of the phase and group velocities will depend on the branch of the solution of (4.13) that is taken, as well as the values of the constants γ, τ. Although no solution of these equations exists in closed form the equations (4.10) and (4.13) can be solved numerically to give corresponding values of r_1, r_2, p, and n, whence c and V can be evaluated from the equations (4.14) and (4.15). If we devote our attention to the lowest mode in which $0 \leqslant r_1 h < \frac{1}{2}\pi$ and $\frac{1}{2}\pi < r_2 h \leqslant \pi$, we see that as r_1 tends to zero, hr_2 tends to π, c to $(\gamma)^{\frac{1}{2}}c_2$, and ϕr_1 to $\pi/2h\tau$, so that V tends to $(\gamma)^{\frac{1}{2}}(1+2\tau)c_2/(\gamma+2\tau)$. In addition, if $r_1 = 0$,

$$(hp)^2 = \pi^2/(\gamma-1),$$

and this expression gives the maximum value of p in this mode. As r_1 increases we see from (4.14) that the largest possible value of r_1 occurs when $r_2^2 = \gamma r_1^2$, in which case $p = 0$, $c = \infty$, and $V = 0$. These features are illustrated in Fig. 1, where computed values of c and V are plotted against values of ph, the constants γ, τ being given the values of $1\frac{1}{2}$, 1 respectively. The most interesting feature of the graph is that the group velocity is found to have a maximum of $1{\cdot}06c_2$, when ph is approximately 4. Jeffreys [(5), pp. 515–18] has shown that when the group velocity has a stationary value, the dispersion will be comparatively small and that comparatively large amplitudes will be found at a large distance from the source. Bearing this in mind, in this particular case we would expect the front of the wave to travel with a velocity of $1{\cdot}06c_2$, with wave-lengths near the value given by $ph = 4$. The longer waves would arrive later and would have comparatively small amplitudes.

(ii) *Case* $c_1 \leqslant c \leqslant c_2$. In this case we take

$$r_1^2 = p^2-n^2/c_1^2, \qquad r_2^2 = n^2/c_2^2-p^2, \tag{4.17}$$

r_1 and r_2 being real and positive. Here the general solutions of the equations (4.8) and (4.9) are

$$\Psi_1 = A\cosh r_1 z+B\sinh r_1 z, \tag{4.18}$$

and $$\Psi_2 = C\cos r_2 z+D\sin r_2 z. \tag{4.19}$$

If these equations are substituted in the boundary conditions we obtain the equation

$$\tau r_1 \tanh r_1 h = r_2 \tan r_2 h, \tag{4.20}$$

which again is an equation for n in terms of p, the solution possessing an infinite number of branches. As the left-hand side of (4.20) is always positive, r_2 in the mth mode must lie in the interval

$$(m-1)\pi \leqslant hr_2 \leqslant (m-\tfrac{1}{2})\pi.$$

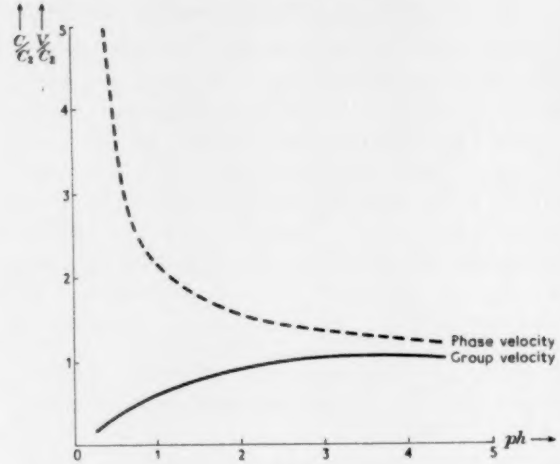

FIG. 1

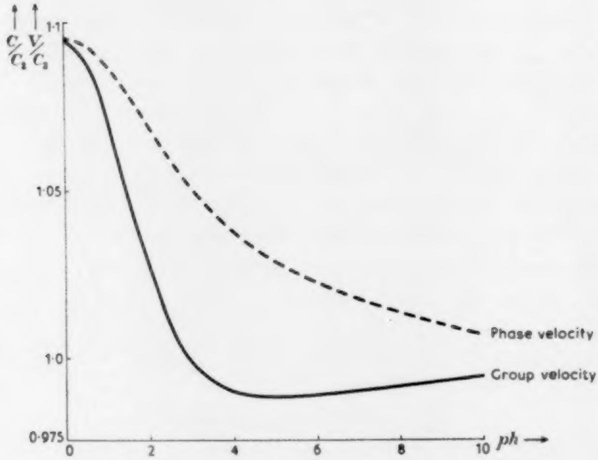

FIG. 2

From the equations (4.17) we see that as hr_2 tends to $(m-1)\pi$, r_1 tends to zero, so that p tends to a lower limit given by $p^2 = (m-1)^2\pi^2/(\gamma-1)h^2$ while the minimum 'cut-off' frequency is given by

$$n^2 = \gamma(m-1)^2\pi^2c_2/(\gamma-1)h^2, \tag{4.21}$$

for the mth mode of vibration. When hr_2 tends to the upper limit $(m-\frac{1}{2})\pi$,

both n and p become very large. From (4.17) the phase velocity c can be expressed in the form

$$c^2 = \gamma(r_1^2+r_2^2)c_2^2/(\gamma r_1^2+r_2^2). \tag{4.22}$$

The group velocity V is again given by the expression (4.15), but in this case

$$\tau\phi = (\tan hr_2+hr_2\sec^2 hr_2)/(\tanh hr_1+hr_1\operatorname{sech}^2 hr_1). \tag{4.23}$$

In all modes except the first we see that if $hr_2 = (m-1)\pi$ then $r_1 = 0$, $c = \gamma^{\frac{1}{2}}c_2$, $\tau\phi r_1 = \frac{1}{2}r_2$ so that $V = (1+2\tau)\gamma^{\frac{1}{2}}c_2/(\gamma+2\tau)$, whilst if $hr_2 = (m-\frac{1}{2})\pi$, r_1 is infinite, so that both c and V are equal to c_2. Thus all the modes above the first will have similar properties, which will depend on the values of the constants γ, τ. The first mode is exceptional because, from (4.20), $r_2 = \tau r_1$ when r_2 is very small. From this it can be shown that as r_2 tends to zero the limits of c and V are both $[\gamma(1+\tau^2)/(\gamma+\tau^2)]^{\frac{1}{2}}c_2$, while, as hr_2 tends to the upper limit $\frac{1}{2}\pi$, c and V tend to the limit c_2. The properties of the first mode are illustrated in Fig. 2 by graphs of c and V against ph, where, for the purpose of computation, the values of γ, τ have again been taken as 1·5, 1. The main features are the maximum at $p = 0$, when $V = c = 1{\cdot}095c_2$, and the minimum group velocity around $ph = 5$, where $V = 0{\cdot}989c_2$.

(iii) *Case* $c \leqslant c_2$. Here the solutions for Ψ_1 and Ψ_2 are hyperbolic functions, which lead to the equation

$$\tau r_1 \tanh r_1 h = -r_2 \tanh r_2 h, \tag{4.24}$$

when combined with the boundary conditions. A glance at this equation shows that it cannot be satisfied by real values of r_1 and r_2. This means that no wave can travel along the plate with a phase velocity less than c_2.

Thus transverse waves of two kinds can be transmitted along a double plate. If the thickness $2h$ of the plate is large compared with the lengths of the waves, then it can be seen from the equations (4.6), (4.7), (4.11), (4.12), (4.18), and (4.19) that the waves of the first kind would correspond to transverse waves travelling through the material and being refracted and reflected at the face $z = 0$, as well as reflected at the faces $z = \mp h$. The waves of the second kind now correspond to transverse waves travelling in the material $z > 0$, being reflected back at $z = h$ and 'totally reflected' at $z = 0$, with the vibration only penetrating a very short distance into the other medium.

5. Love waves in a surface layer

We consider a surface layer $0 < z < h$, in which the velocity of rotational waves is c_2, resting on a semi-infinite material which occupies the space $z < 0$ and in which the velocity of rotational waves is c_1.

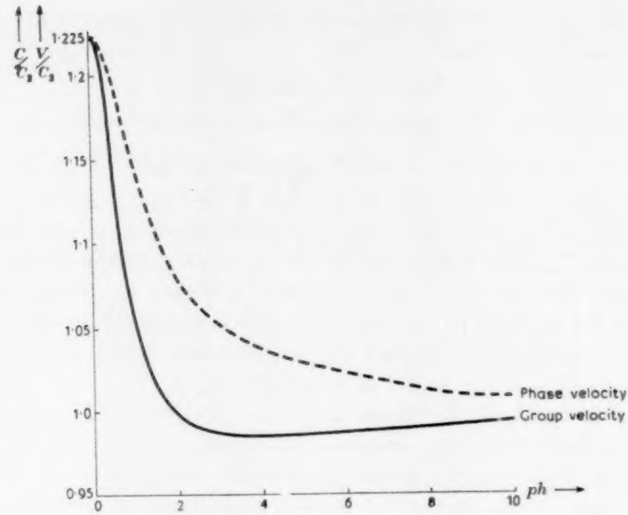

FIG. 3

Here the equations of motion are the same as in (4.1) and (4.2), and we take ψ_1 and ψ_2 in the form

$$\psi_2 = (A\cos r_2 z + B\sin r_2 z)e^{i(px-nt)}, \quad \text{in } 0 < z < h, \tag{5.1}$$

and

$$\psi_1 = Ce^{r_1 z + i(px-nt)}, \quad \text{in } z < 0, \tag{5.2}$$

where r_1 and r_2 satisfy the equations

$$r_1^2 = p^2 - n^2/c_1^2 \quad \text{and} \quad r_2^2 = n^2/c_2^2 - p^2, \tag{5.3}$$

so that

$$\left.\begin{aligned} n^2 &= \gamma c_2^2(r_1^2+r_2^2)/(\gamma-1) \\ p^2 &= (\gamma r_1^2 + r_2^2)/(\gamma-1) \end{aligned}\right\}, \tag{5.4}$$

where $\gamma = c_1^2/c_2^2$.

The boundary conditions are $\partial\psi_2/\partial z = 0$ on $z = h$, while $\psi_1 = \psi_2$ and $\tau\,\partial\psi_1/\partial z = \partial\psi_2/\partial z$ on $z = 0$. Substituting the expressions for ψ_1 and ψ_2 from (5.1) and (5.2) in the boundary conditions gives the equation

$$\tau r_1 = r_2 \tan r_2 h, \tag{5.5}$$

first derived by Love (**2**). This may be regarded as an equation for n in terms of p with the solution possessing an infinite number of branches.

In the mth mode r_2 lies in the interval given by the expression

$$(m-1)\pi \leqslant hr_2 \leqslant (m-\tfrac{1}{2})\pi,$$

and a glance at (5.4) shows that in this mode

$$(hp)^2 \geqslant (m-1)^2\pi^2/(\gamma-1), \tag{5.6}$$

and

$$(hn)^2 \geqslant \gamma c_2^2(m-1)^2\pi^2/(\gamma-1), \tag{5.7}$$

the latter inequality yielding the cut-off frequency for each mode. The phase velocity c may be obtained from the equations (5.4) in the form

$$c^2 = \gamma c_2^2(r_1^2+r_2^2)/(\gamma r_1^2+r_2^2), \tag{5.8}$$

and if we differentiate (5.3) and (5.5) we obtain the group velocity

$$V = \frac{[\tau(r_1^2+r_2^2)+hr_1(r_2^2+\tau^2 r_1^2)]\gamma c_2^2}{[\tau(\gamma r_1^2+r_2^2)+\gamma hr_1(r_2^2+\tau^2 r_1^2)]c}. \tag{5.9}$$

If we look at the mth mode we see that if hr_2 is at the lower limit $(m-1)\pi$ then $r_1 = 0$, and both c and V tend to the upper limit c_1. In particular, we see from (5.6) that the longest waves only occur in the first mode, in which we see from (5.5) that $\tau r_1 = hr_2^2$ as r_2 tends to zero. Using this result in (5.8) and (5.9) we see that for small r_2

$$c = \gamma^{\frac{1}{2}}c_2\left[1-\frac{h^2}{2\tau^2}(\gamma-1)r_2^2+O(r_2^4)\right], \tag{5.10}$$

and

$$V = \gamma^{\frac{1}{2}}c_2\left[1-\frac{3h^2}{2\tau^2}(\gamma-1)r_2^2+O(r_2^4)\right]. \tag{5.11}$$

We see from (5.10) and (5.11) that both c and V have a maximum value of c_1 when $r_2 = 0$, corresponding to $p = 0$. When hr_2 approaches an upper limit $(m-\frac{1}{2})\pi$, the value of r_1 becomes very large, corresponding to short waves of high frequency. For large values of r_1 we obtain from (5.8) the result that

$$c = c_2[1+(\gamma-1)r_2^2/2\gamma r_1^2+O(r_1^{-4})], \tag{5.12}$$

while from the expressions (5.9) and (5.12)

$$V = c_2[1-(\gamma-1)r_2^2/2\gamma r_1^2+O(r_1^{-3})]. \tag{5.13}$$

Remembering our assumption that $\gamma > 1$, we see that as p tends to infinity, c tends to the limit c_2 from above, but V tends to the limit c_2 from below. In any one mode, c and V are continuous functions of p, so that V must possess a minimum which is less than c_2. Denoting by $\bar{V}_m$ the minimum group velocity in the mth mode, we see that the group velocity must lie between c_1 and $\bar{V}_m$. Thus for any mode the general shape of the dispersion curves will be as in Fig. 3, which illustrates the behaviour of c and V in the first mode with the constants γ and τ having the value $1\frac{1}{2}$ and 1 respectively. The existence of a minimum group velocity was first demonstrated by Jeffreys (**1**, **3**).

By inspection of (5.9),

$$V \geqslant c_2^2/c \geqslant c_2^2/c_1. \tag{5.14}$$

As m becomes large it can be shown from (5.9) that V tends to the limit c_2^2/c except in the immediate region of $r_1 = 0$. Thus, in the higher modes it is to be expected that the minimum group velocity is only a little more

than c_2^2/c_1, and occurs near $r_2 = (m-1)\pi/h$. In addition, if, in (5.9) τ is made large, then V will again approach c_2^2/c, whilst if γ increases, so does c_1, as will c for some particular value of r_2 and r_1. Consequently it is to be expected that $\bar{V}_m$ will decrease when m, γ, or τ increase. However, in every case, c_2^2/c_1 will be a lower bound for V.

In the case $\gamma = 1\frac{1}{2}$, $c_2^2/c_1 = 0{\cdot}316c_2$. Taking $\tau = 1$, it was found that $\bar{V}_m$ was $0{\cdot}986c_2$, $0{\cdot}94c_2$, $0{\cdot}91c_2$, and $0{\cdot}84c_2$ for $m = 1, 2, 3$, and 10 respectively. If $\gamma = 5$, $\bar{V}_1$ was found to be $0{\cdot}96c_2$, $0{\cdot}98c_2$, $0{\cdot}86c_2$, and $0{\cdot}85c_2$ for $\tau = 1, 2, 3$, and 5 respectively. The computations were done with the aid of an 'autocode' programme for the Manchester University 'Pegasus' computer and bear out the deduction that $\bar{V}_m$ decreases for increases in m, γ, or τ.

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LOVE WAVES IN HYPOELASTIC BODY OF GRADE ZERO

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[Received 31 October 1957]

SUMMARY

The propagation of Love waves in the hypoelastic body of grade zero has been considered. It is found that the wave velocity retains the same character as that in the classical linearly elastic body. The difference lies in the fact that one of the normal stresses is not identically zero in the hypoelastic body considered, whereas all the normal stresses are zero in the classical elastic body.

1. Introduction

THE theory of elastic response based on time rates, called *hypoelasticity* by Truesdell (**1**, **2**, **3**), has been illustrated by him and Green (**4**) in some simple problems. Thus the problems of simple extension, pure shear, hydrostatic pressure, and others reveal interesting phenomena characteristic of the theory. Green (**5**, **6**) has also studied the theory for both incompressible and compressible bodies with reference to the theory of plasticity. In this paper, the propagation of Love waves in a particular type of hypoelasticity, that of grade zero, has been examined. It is found that the wave velocity possesses a feature similar to that in the classical linearly elastic body. There is, however, a difference; whereas in the classical elasticity all the normal stress components are identically zero, one of the normal components in hypoelasticity considered does not vanish identically.

2. Fundamental equations

The isotropic hypoelastic body of grade zero is defined by the constitutive equations (**1**, **2**, **3**, **4**)

$$\mathring{s}^{ik}g_{kj} = \frac{\nu}{1-2\nu}d^k_k\,\delta^i_j+d^i_j, \tag{1}$$

where

$$\mathring{s}^{ik} = \frac{\partial s^{ik}}{\partial t}+v^m s^{ik}_{,m}-s^{im}v^k_{,m}-s^{mk}v^i_{,m}+s^{ik}d^m_m, \tag{2}$$

$$d_{ij} = \tfrac{1}{2}(v_{i,j}+v_{j,i}). \tag{3}$$

In the above equations g_{ik} is the covariant metric tensor of a general fixed system of coordinates, v_i is the covariant velocity vector, $2\mu s^i_j$ is the stress tensor (μ being the rigidity modulus of classical elasticity), ν is the classical Poisson ratio, t denotes time, and a comma denotes covariant

differentiation. The other differential equations of the theory are the equations of motion and continuity written in the forms

$$2\mu s^{ij}_{,j} = \rho\left(\frac{\partial v^i}{\partial t} + v^i_{,j}\, v^j\right), \tag{4}$$

$$\frac{\partial \rho}{\partial t} + (\rho v^i)_{,i} = 0, \tag{5}$$

where ρ is the density and body forces are taken as zero.

3. Love waves

We consider a semi-infinite medium bounded by the plane $z = 0$, and the positive direction of the z-axis is taken into the medium. A layer of thickness T of different material is spread over the surface $z = 0$ so that the upper surface of the layer is given by $z = -T$. Let (μ, ρ) be the rigidity modulus and density respectively of the layer and (μ', ρ') be the corresponding values for the material below. We consider the possibility of the propagation of a purely transverse wave (of Love type) in the medium such that the disturbance penetrates only a little distance into the interior. Let the wave be propagated parallel to the x-axis with velocity c. We assume that the velocity components are

$$v^1 = v^3 = 0, \tag{6}$$

$$v^2 \equiv v_y = V(z)\exp\{ik(x-ct)\}, \tag{7}$$

where V is a function of z only and k measures the wavelength, and where $i \equiv \sqrt{(-1)}$. On the right-hand side of the equation (7), only the real part is to be taken. The equation (5) is satisfied if

$$\rho = \text{constant}. \tag{8}$$

Then using the relations (2) and (3) equation (1) reduces to

$$\left.\begin{aligned} \frac{\partial s_{xx}}{\partial t} + v_y \frac{\partial s_{xx}}{\partial y} = 0, \quad \frac{\partial s_{zz}}{\partial t} + v_y \frac{\partial s_{zz}}{\partial y} = 0, \quad \frac{\partial s_{xz}}{\partial t} + v_y \frac{\partial s_{xz}}{\partial y} = 0 \\ \frac{\partial s_{yy}}{\partial t} + v_y \frac{\partial s_{yy}}{\partial y} - 2\left(s_{xy}\frac{\partial v_y}{\partial x} + s_{yz}\frac{\partial v_y}{\partial z}\right) = 0 \\ \frac{\partial s_{xy}}{\partial t} + v_y \frac{\partial s_{xy}}{\partial y} - \left(s_{xx}\frac{\partial v_y}{\partial x} + s_{xz}\frac{\partial v_y}{\partial z}\right) = \frac{1}{2}\frac{\partial v_y}{\partial x} \\ \frac{\partial s_{yz}}{\partial t} + v_y \frac{\partial s_{yz}}{\partial y} - \left(s_{xz}\frac{\partial v_y}{\partial x} + s_{zz}\frac{\partial v_y}{\partial z}\right) = \frac{1}{2}\frac{\partial v_y}{\partial z} \end{aligned}\right\}, \tag{9}$$

where we have written $s_{11} = s_{xx}$, $s_{12} = s_{xy}, \ldots$, etc.

The first three equations of (9) are satisfied if

$$s_{xx} = s_{zz} = s_{xz} = 0. \tag{10}$$

The solution of the last two equations of (9), when relations (10) are used, are

$$\left.\begin{aligned} s_{xy} &= -\frac{1}{2c} V \exp\{ik(x-ct)\} \\ s_{yz} &= \frac{i}{2kc}\frac{dV}{dz}\exp\{ik(x-ct)\} \end{aligned}\right\}. \tag{11}$$

The fourth equation of (9) is then satisfied if

$$s_{yy} = \frac{1}{2c^2}\left[V^2 - \frac{1}{k^2}\left(\frac{dV}{dz}\right)^2\right]\exp\{2ik(x-ct)\}. \tag{12}$$

Two of the equations (4) are identically satisfied. The other equation gives

$$\frac{d^2V}{dr^2} + k^2\left(\frac{c^2}{\beta^2} - 1\right)V = 0, \tag{13}$$

where

$$\beta^2 = \frac{\mu}{\rho}. \tag{14}$$

The solution of (13) can be written as

$$V = A\cos(sz) + B\sin(sz) \quad (-T \leqslant z \leqslant 0), \tag{15}$$

$$V = A\exp(-s'z) \qquad (0 \leqslant z \leqslant \infty), \tag{16}$$

where s' is a positive real constant. The condition that the displacement is continuous across $z = 0$ has been utilized in writing the equations (15) and (16). Equations (13) and (15) are satisfied if

$$s^2 = \frac{k^2}{\beta^2}(c^2 - \beta^2). \tag{17}$$

In (13) replacing β^2 by β'^2 where

$$\beta'^2 = \frac{\mu'}{\rho'}, \tag{18}$$

and then using (16) we have

$$s' = +\frac{k}{\beta'}(\beta'^2 - c^2)^{\frac{1}{2}}. \tag{19}$$

Since s' is assumed to be real and positive,

$$\beta' > c. \tag{20}$$

The condition that the stress $2\mu s_{yz}$ should be continuous across $z = 0$ gives, by (11), (15), and (16),

$$-sB\mu = s'A\mu'. \tag{21}$$

Again the condition that $2\mu s_{yz} = 0$ at $z = -T$ gives

$$A\sin(sT) + B\cos(sT) = 0. \tag{22}$$

From (21) and (22), $$s\mu\tan(sT) = s'\mu'. \tag{23}$$

Since the right-hand side of (23) is real and positive, it is concluded from this and (17) that
$$c > \beta. \tag{24}$$

Thus from (20) and (24), $$\beta < c < \beta'. \tag{25}$$

The result (25) is well known in the classical theory of linear elasticity (**7**, **8**). From (15), (21), and (23),
$$V = A\cos\{s(z+T)\}/\cos(sT) \quad (-T \leqslant z \leqslant 0), \tag{26}$$

where $v_y = A$ at $x = z = 0$, $t = 0$. From (12), (16), and (26), we have, using (17) and (19),
$$2\mu s_{yy} = A^2\mu\{c^2\beta^2\cos^2(sT)\}^{-1}[\beta^2 - c^2\sin^2\{s(z+T)\}\exp\{2ik(x-ct)\}] \quad (-T \leqslant z \leqslant 0), \tag{27}$$
$$2\mu' s_{yy} = A^2\mu'\beta'^{-2}\exp\{2ik(x-ct)-2\beta' z\} \quad (0 \leqslant z \leqslant \infty). \tag{28}$$

From (27) and (28) it is seen that one of the normal stresses is not identically zero for the hypoelastic body of grade zero, whereas in the classical linear elasticity all the normal stresses are zero identically.

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THE QUARTERLY JOURNAL OF MECHANICS
AND APPLIED MATHEMATICS

VOLUME XI PART 4 NOVEMBER 1958

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The Editorial Board gratefully acknowledge the support given by: Blackburn & General Aircraft Limited; Bristol Aeroplane Company; Courtaulds Scientific and Educational Trust Fund; English Electric Company; Hawker Siddeley Group Limited; Imperial Chemical Industries Limited; Metropolitan-Vickers Electrical Company Limited; The Shell Petroleum Co. Limited; Vickers-Armstrongs (Aircraft) Limited.